5 Sectional Curvature.

Throughout this section, we let (M, g) an m-dimensional (pseudo) Riemannian manifold, and ∇ the covariant derivative defined in (4.14).

Tensors. A correspondence $S : M \ni \mathrm{P} \mapsto S_{\mathrm{P}}$ of a point P and a multi-linear map $S_{\mathrm{P}} : (T_{\mathrm{P}}M)^k \to \mathbb{R}$ is called a (k-th order covariant) *tensor field* or a *tensor* on M. For such a tensor field S trivially induces a map

$$\hat{S} : \bigl(\mathfrak{X}(M)\bigr)^p \ni (X_1, \dots, X_p) \mapsto S(X_1, \dots, X_p) \in \mathcal{F}(M) \tag{5.1}$$

where $\mathfrak{X}(M)$ is the set ($C^\infty(M)$-module) of C^∞-vector fields of M, and $\mathcal{F}(M)$ is the set of real-valued function on M. The tensor field S is said to be *smooth* of *class* C^∞ if $\hat{S}S(X_1, \dots, X_p)$ as in (5.1) is of class C^∞ for an arbitrary $X_1, \dots, X_p$.

Example 5.1. The (pseudo) Riemannian metric g is a smooth tensor field.

Example 5.2. A smooth 1-form on M is a smooth, first-order covariant tensor field on M.

We denote by

$$\Gamma\,(\otimes^p T^*M) = \Gamma(T^*M \otimes \dots \otimes T^*M) \tag{5.2}$$

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the set of p-th order covariant tensor fields on M. [7] The set $\Gamma\,(\otimes^p T^*M)$ is a vector space over $\mathbb{R}$. Moreover, for each $S \in \Gamma(\otimes^p T^*M)$ and $f \in C^\infty(M)$, $fS := (\mathrm{P} \mapsto f(\mathrm{P})S_{\mathrm{P}})$ is also an element of $\Gamma(\otimes^p T^*M)$. That is, $\Gamma(\otimes^p T^*M)$ is a $C^\infty(M)$-module.

Lemma 5.3. *Let S be a p-th covariant tensor field on M and $\hat{S} : \mathfrak{X}(M)^p \to C^\infty(M)$ the map induced by S as in* (5.1) *Then for an arbitrary $f \in C^\infty(M)$, it holds that*

$$\hat{S}(X_1, \dots, fX_j, \dots, X_p) = f\hat{S}(X_1, \dots, X_j, \dots, X_p), \tag{5.3}$$

where $X_1, \dots, X_p \in \mathfrak{X}(M)$.

Proof. For each $\mathrm{P} \in M$,

$$\begin{aligned}
&S(X_1, \dots, fX_j, \dots, X_p)(\mathrm{P}) \\
&\qquad = S_{\mathrm{P}}\bigl((X_1)_{\mathrm{P}}, \dots, f(\mathrm{P})(X_j)_{\mathrm{P}}, \dots, (X_p)_{\mathrm{P}}\bigr) \\
&\qquad = f(\mathrm{P})S_{\mathrm{P}}\bigl((X_1)_{\mathrm{P}}, \dots, (X_j)_{\mathrm{P}}, \dots, (X_p)_{\mathrm{P}}\bigr) \\
&\qquad = (fS)(X_1, \dots, X_p)(\mathrm{P}). \qquad \square
\end{aligned}$$

Proposition 5.4. *A multi-linear map $\hat{S} : \bigl(\mathfrak{X}(M)\bigr)^p \to C^\infty(M)$ is induced from a certain $S \in \Gamma(\otimes^p T^*M)$ as in* (5.1) *if $\hat{S}$ is $C^\infty(M)$-multi-linear, that is,* (5.3) *holds for any $f \in C^\infty(M)$ and $X_1, \dots, X_p \in \mathfrak{X}(M)$.*

[7] The symbol "$\otimes$" in (5.2) means the *tensor product*. For example, $T^*M \otimes T^*M$ is a tensor product of the cotangent bundles, which is a certain vector bundle over M. The notion $\Gamma(*)$ means the set of sections of the vector bundle "$*$". Anyway, we do not give a precise meaning of these notations.

Proof. Assume that (5.1) holds. We fix $\mathrm{P} \in M$ and take a local coordinate system $(U; u^1, \dots, u^m)$ of M around P. According to this coordinate system, we set $X_j = \sum_{l=1}^m \xi_j^l (\partial/\partial u^l)$ $(j = 1, \dots, m)$, where ξ_j^l's are C^∞-functions on U. Then by (5.1),

$$\hat{S}(X_1, \dots, X_p) = \sum \xi_1^{i_1} \dots \xi_p^{i_p} \hat{S}\left(\frac{\partial}{\partial u^{i_1}}, \dots, \frac{\partial}{\partial u^{i_p}}\right)$$

holds, where the sum in the right-hand side is taken over $i_j = 1, \dots, m$ $(j = 1, \dots, p)$. This means the value of the left-hand side at P is determined by $\xi_j^{i_l}(\mathrm{P})$, which depend only on $(X_j)_{\mathrm{P}}$. Hence, for each $\boldsymbol{v}_1, \dots, \boldsymbol{v}_p \in T_{\mathrm{P}}M$, we can define

$$S_{\mathrm{P}}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_p) := \hat{S}(X_1, \dots, X_m),$$

where X_j is an arbitrary vector field on M such that $X_j(\mathrm{P}) = \boldsymbol{v}_j$. Then $S\colon \mathrm{P} \mapsto S_{\mathrm{P}}$ is the desired one. □

If $\hat{S}$ is induced from a tensor field S, we say that $\hat{S}$ itself is a tensor field. From now on, we denote $\hat{S}$ in (5.1) by S for a simplicity. Then, for each $S \in \Gamma(\otimes^p T^* M)$, the C^∞-multi-linear map

$$S\colon \bigl(\mathfrak{X}(M)\bigr)^p \longrightarrow C^\infty(M) \tag{5.4}$$

is induced.

Taking a local coordinate system $(U; u^1, \dots, u^m)$ on M, we set

$$S_{i_1,\dots,i_p} := S\left(\frac{\partial}{\partial u^{i_1}}, \dots \frac{\partial}{\partial u^{i_p}}\right) \qquad (i_1, \dots, i_p = 1, \dots, m), \tag{5.5}$$

which are called the *components* of S with respect to the local coordinate system (u^j). Let $\{S_{a_1,\dots,a_p}\}$ be the components of S with respect to another coordinate system (x^a). Then it holds that

$$S_{a_1,\dots,a_p} = \frac{\partial u^{i_1}}{\partial x^{a^1}} \dots \frac{\partial u^{i_p}}{\partial x^{a^p}} S_{i_1,\dots,i_p}. \tag{5.6}$$

Proposition 5.5. *Let $S\colon \bigl(\mathfrak{X}(M)\bigr)^p \to C^\infty(M)$ be a multi-linear map, and set $S_{i_1,\dots,i_p}$ by* (5.5)*. Then S is a tensor field on M if and only if it satisfy* (5.6) *for an arbitrary coordinate change $(x^a) \mapsto (u^j)$.*

Proof. Problem 5-1. □

The Curvature Tensor. Let ∇ be the covariant derivative on (M, g), as defined in (4.14), which is considered as

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \longmapsto \nabla_X Y \in \mathfrak{X}(M).$$

Remark 5.6. The tri-linear map

$$D\colon \bigl(\mathfrak{X}(M)\bigr)^3 \ni X, Y, Z \mapsto g(\nabla_X Y, Z) \in C^\infty(M)$$

is not a tensor field. In fact, (4.16) means that $D(X, fY, Z)$ and $fD(X, Y, Z)$ may not coincide. But for a fixed $Y \in \mathfrak{X}(M)$,

$$DY\colon \mathfrak{X}(M) \times \mathfrak{X}(M)(X, Z) \mapsto g(\nabla_X Y, Z) \in C^\infty(M)$$

is a tensor because of (4.15).

As seen in the proof in Proposition 4.10, the 4-linear map $R\colon \bigl(\mathfrak{X}(M)\bigr)^4 \to C^\infty(M)$ defined by

(5.7) $R(X,Y,Z,W) := g\bigl(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W\bigr),$

where $[X,Y]$ denotes the *Lie bracket* of the vector fields, is a tensor field, which we call the *curvature tensor*, or the *Riemann-Christoffel curvature tensor* of (M,g).

Proposition 5.7. *The curvature tensor R has the following symmetricity:*

(1) $R(Y,X,Z,W) = -R(X,Y,Z,W)$.

(2) $R(X,Y,W,Z) = -R(X,Y,Z,W)$.

(3) $R(X,Y,W,Z) = R(W,Z,X,Y)$.

(4) $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$,

where X, Y, Z and W are vector fields.

Proof. The equality (1) follows from the property of the Lie bracket $[Y,X] = -[X,Y]$. The equality (4) can be proved by the property (4.17) and the Jacobi identity

$$\bigl[[X,Y],Z\bigr] + \bigl[[Y,Z],X\bigr] + \bigl[[Z,X],Y\bigr] = 0$$

for the Lie bracket. The property (2) can be shown by applying (4.18) and (4.17) (Problem 5-2). The property (3) follows from

(4), (1) and (2). In fact, summing up

$$\begin{aligned}
R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) &= 0,\\
R(Y,Z,W,X) + R(Z,W,Y,X) + R(W,Y,Z,X) &= 0,\\
R(Z,W,X,Y) + R(W,X,Z,Y) + R(X,Z,W,Y) &= 0,\\
R(W,X,Y,Z) + R(X,Y,W,Z) + R(Y,W,X,Z) &= 0,
\end{aligned}$$

(3) follows. □

Proposition 5.8. *Assume two tensors R_1 and $R_2 \in \Gamma(\otimes^4 T^*M)$ satisfy the symmetricity as in Proposition 5.7. If*

$$R_1(X,Y,Y,X) = R_2(X,Y,Y,X)$$

holds for all X, $Y \in \mathfrak{X}(M)$, then $R_1 = R_2$

Proof. Expanding

$$\begin{aligned}
&R_1(X+sZ, Y+tW, Y+tW, X+sZ)\\
&\qquad = R_2(X+sZ, Y+tW, Y+tW, X+sZ),
\end{aligned}$$

we have the conclusion from the coefficients of st. □

Sectional Curvature.

Lemma 5.9. *Let R be the curvature tensor of a (pseudo) Riemannian manifold (M,g). Then, for each $\mathrm{P} \in M$,*

$$\frac{R(\boldsymbol{v},\boldsymbol{w},\boldsymbol{w},\boldsymbol{v})}{g(\boldsymbol{v},\boldsymbol{v})g(\boldsymbol{w},\boldsymbol{w}) - g(\boldsymbol{v},\boldsymbol{w})^2} \qquad (\boldsymbol{v},\boldsymbol{w} \in T_{\mathrm{P}}M)$$

depends only on the 2*-dimensional subspace of* $T_{\mathrm{P}}M$ *spanned by* $\{\boldsymbol{v}, \boldsymbol{w}\}$ *whenever*

$$g(\boldsymbol{v},\boldsymbol{v})g(\boldsymbol{w},\boldsymbol{w}) - g(\boldsymbol{v},\boldsymbol{w})^2 \neq 0. \tag{5.8}$$

Proof. Set

$$(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{v},\boldsymbol{w})A,$$

where $A \in \mathrm{GL}(2,\mathbb{R})$. Then

$$\begin{aligned} R(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y},\boldsymbol{x}) &= \det A R(\boldsymbol{v},\boldsymbol{w},\boldsymbol{w},\boldsymbol{v}),\\ g(\boldsymbol{x},\boldsymbol{x})g(\boldsymbol{y},\boldsymbol{y}) - g(\boldsymbol{x},\boldsymbol{y})^2 &= \det A\big(g(\boldsymbol{v},\boldsymbol{v})g(\boldsymbol{w},\boldsymbol{w}) - g(\boldsymbol{v},\boldsymbol{w})^2\big). \end{aligned}$$

Hence the conclusion follows. □

Remark 5.10. When g is positive definite (i.e., (M,g) is a Riemannian manifold), (5.8) holds if and only if $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent. On the other hand, when g is indefinite, the left-hand side of (5.8) may vanish even if $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent. In this case, (5.8) holds if and only if $\mathrm{Span}\{\boldsymbol{v},\boldsymbol{w}\}$ is a non-degenerate subspace of $T_{\mathrm{P}}M$.

Definition 5.11. For a 2-dimensional non-degenerate subspace $\Pi_{\mathrm{P}} \subset T_{\mathrm{P}}M$, we set

$$K(\Pi_{\mathrm{P}}) := \frac{R(\boldsymbol{v},\boldsymbol{w},\boldsymbol{w},\boldsymbol{v})}{g(\boldsymbol{v},\boldsymbol{v})g(\boldsymbol{w},\boldsymbol{w}) - g(\boldsymbol{v},\boldsymbol{w})^2)},$$

where $\{\boldsymbol{v},\boldsymbol{w}\}$ is a basis of Π_{P}. We call it the *sectional curvature* at Π_{P}.

Remark 5.12. The set of 2-dimensional subspaces on an n-dimensional vector space V can be endowed with the structure of a compact $(2n-3)$-dimensional manifold, denoted by $\mathrm{Gr}_2(V)$, which is called the 2-*Grassmanian manifold* over V. So, when (M,g) is a Riemannian, the sectional curvature can be considered as a smooth map

$$K\colon \mathrm{Gr}_2(TM) := \bigcup_{\mathrm{P}\in M} \mathrm{Gr}_2(T_{\mathrm{P}}M) \to \mathbb{R}.$$

Example 5.13. Let (M,g) be a 2-dimensional Riemannian manifold. Since $\mathrm{Gr}_2(T_{\mathrm{P}}M)$ consists of one point, the sectional curvature K can be regarded as a function defined on M itself. In this case, the sectional curvature is written as

$$\begin{aligned} K = {} & \frac{E(E_vG_v - 2F_uG_v + G_u^2)}{4(EG-F^2)^2} \\ & + \frac{F(E_uG_v - E_vG_u - 2E_vF_v - 2F_uG_u + 4F_uF_v)}{4(EG-F^2)^2} \\ & + \frac{G(E_uG_u - 2E_uF_v + E_v^2)}{4(EG-F^2)^2} - \frac{E_{vv} - 2F_{uv} + G_{uu}}{2(EG-F^2)}, \end{aligned}$$

where $(u^1,u^2) = (u,v)$ is a local coordinate system and

$$E = g_{11}, \qquad F = g_{12} = g_{21}, \qquad G = g_{22}.$$

Constant Sectional Curvature. A Riemannian manifold (M,g) is said to be a *space of constant sectional curvature* if K is constant everywhere.

Proposition 5.14. *A (pseudo) Riemannian manifold (M, g) has constant sectional curvature k if and only if its curvature tensor R satisfies*

$$R(X, Y, Z, W) = k(g(X, T)g(Y, Z) - g(X, Z)g(Y, T)). \tag{5.9}$$

Proof. If R satisfy (5.9), $K = k$ is constant obviously. Conversely, assume $K = k$ is constant. Then (5.9) holds for $Z = Y$, $W = X$. Since the right-hand side has the symmetric property as in Proposition 5.7, Proposition 5.8 yields (5.9). □

Example 5.15. The curvature tensor of the Euclidean space $\mathbb{R}^n$ vanishes identically, because R vanishes identically. The covariant derivative of $\mathbb{R}^n$ is identified with the directional derivative D. This means that

$$D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z = O$$

holds for vector fields X, Y and Z.

Example 5.16. Let $k < 0$ and

$$S^n(k) := \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1} \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \frac{1}{k} \right\}.$$

As seen in Example 3.3, this is an n-dimensional submanifold of $\mathbb{R}^{n+1}$, and then is a Riemannian manifold with the induced metric from $\mathbb{R}^{n+1}$.

We compute the sectional curvature $S^n(k)$: The unit normal vector of $S^n(k)$ at $\boldsymbol{x}$ is $\boldsymbol{n} := \boldsymbol{x}/\sqrt{k}$. Then, regarding vector field on $S^n(k)$ as a vector field of $\mathbb{R}^{n+1}$ along $S^n(k)$, we have by Proposition 4.8

$$\begin{aligned}
\nabla_Y Z = [D_Y Z]^{\mathrm{T}} &= D_Y Z - \frac{1}{\sqrt{k}} \langle D_Y Z, \boldsymbol{x} \rangle \boldsymbol{x} \\
&= D_Y Z - \frac{1}{\sqrt{k}} Y \langle Z, \boldsymbol{x} \rangle \boldsymbol{x} + \frac{1}{\sqrt{k}} \langle Z, D_Y \boldsymbol{x} \rangle \boldsymbol{x} \\
&= D_Y Z - \frac{1}{\sqrt{k}} Y \langle Z, \boldsymbol{x} \rangle \boldsymbol{x} + \frac{1}{\sqrt{k}} \langle Z, Y \rangle \boldsymbol{x} \\
&= D_Y Z + \frac{1}{\sqrt{k}} \langle Z, Y \rangle \boldsymbol{x},
\end{aligned}$$

where we used the relation $D_Y \boldsymbol{x} = Y$. Using this relation, we can show that the curvature tensor R satisfy

$$\begin{aligned}
&R(X, Y, Z, W) \\
&\qquad = \big\langle D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, W \big\rangle \\
&\qquad\qquad + k\big(g(X, W)g(Y, Z) - g(X, Z)g(Y, T)\big).
\end{aligned}$$

Hence by Proposition 5.14, we obtain that the sectional curvature of $S^n(k)$ is k.

Example 5.17. Let k is a positive constant and

$$H^n(-k) := \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1}_1 \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -\frac{1}{k}, x^0 > 0 \right\},$$

where $\boldsymbol{x} = (x^0, \dots, x^n)$. Then $H^n(-k)$ is a space-like hypersurface in the Lorentz-Minkowski space $\mathbb{R}^{n+1}_1$, as seen in Example 3.5, called the *hyperbolic space*. Since $H^n(-k)$ is a space-like hypersurface, the induced metric gives a Riemannian metric, and then $H^n(-k)$ is a Riemannian manifold.

By the completely same method as in the previous example, one can show that $H^n(-k)$ has constant sectional curvature $-k$.

Exercises

5-1 Prove Proposition 5.5.

5-2 Show (3) in Proposition 5.7.

5-3 Compute the sectional curvature of a Riemannian 2-manifold (M, g) with

$$g_{11} = g_{22} = \frac{4}{(1 + k(u^2 + v^2))^2}, \quad g_{12} = g_{21} = 0,$$

where $(u, v) = (u^1, u^2)$ is a local coordinate system.