4 The Curvature Tensor.

Change of Coordinate Systems. Throughout this section, we let (M, g) be a (pseudo) Riemannian m-manifold and take a local coordinate system $(U; u^1, \dots, u^m)$ on a neighborhood of $\mathrm{P} \in U$. Choose another coordinate system $(V; x^1, \dots, x^m)$ on a neighborhood V of P. Then the coordinate change

$$\begin{aligned}(4.1) \qquad \boldsymbol{x} &= (x^1, \dots, x^m) \\ &\mapsto \boldsymbol{u}(\boldsymbol{x}) = (u^1(x^1, \dots, x^m), \dots, u^m(x^1, \dots, x^m))\end{aligned}$$

is defined as a C^∞-map between certain domains in $\mathbb{R}^m$. Since the transformation (4.1) is a diffeomorphism, the inverse

$$\begin{aligned}(4.2) \qquad \boldsymbol{u} &= (u^1, \dots, u^m) \\ &\mapsto \boldsymbol{x}(\boldsymbol{u}) = (x^1(u^1, \dots, u^m), \dots, x^m(u^1, \dots, u^m))\end{aligned}$$

is also C^∞. Thus, the Jacobian matrix

$$J := \begin{pmatrix} \dfrac{\partial u^1}{\partial x^1} & \dots & \dfrac{\partial u^1}{\partial x^m} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial u^m}{\partial x^1} & \dots & \dfrac{\partial u^m}{\partial x^m} \end{pmatrix} = \left(\frac{\partial u^i}{\partial x^a}\right)_{i,a=1,\dots,m}$$

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is invertible on each point of the domain of $\boldsymbol{u}(\boldsymbol{x})$, and the Jacobian matrix of the inverse map $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{u})$ is obtained as

$$\begin{pmatrix} \dfrac{\partial x^1}{\partial u^1} & \dots & \dfrac{\partial x^1}{\partial u^m} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial x^m}{\partial u^1} & \dots & \dfrac{\partial x^m}{\partial u^m} \end{pmatrix} = \left(\frac{\partial x^a}{\partial u^i}\right)_{a,i=1,\dots,m} = J^{-1},$$

where the inverse matrix of the right-hand side is evaluated at $\boldsymbol{u}(\boldsymbol{x})$, that is,

$$(4.3) \qquad \sum_{i=1}^m \frac{\partial x^a}{\partial u^i}\frac{\partial u^i}{\partial x^b} = \delta^a_b, \qquad \text{and} \qquad \sum_{a=1}^m \frac{\partial u^i}{\partial x^a}\frac{\partial x^a}{\partial u^j} = \delta^i_j$$

hold, where δ denotes Kronecker's delta.

Components of Vector fields and Differential forms. Let X be a vector field on M. Then it can be expressed on coordinate neighborhoods $(U; \boldsymbol{u})$ and $(V; \boldsymbol{x})$ as

$$X = \sum_{i=1}^m X^i \frac{\partial}{\partial u^i} = \sum_{a=1}^m \widetilde{X}^a \frac{\partial}{\partial x^a}.$$

Since

$$(4.4) \qquad \frac{\partial}{\partial u^i} = \sum_{a=1}^m \frac{\partial x^a}{\partial u^i}\frac{\partial}{\partial x^a} \quad \text{and} \quad \frac{\partial}{\partial x^a} = \sum_{i=1}^m \frac{\partial u^i}{\partial x^a}\frac{\partial}{\partial u^i},$$

we have the following transformation formula for the components of X:

$$\widetilde{X}^a = \sum_{i=1}^{m} \frac{\partial x^a}{\partial u^i} X^i, \qquad X^i = \sum_{a=1}^{m} \frac{\partial u^i}{\partial x^a} \widetilde{X}^a. \tag{4.5}$$

Thus,

Lemma 4.1. *The components (g_{ij}) and $(\tilde{g}_{ab})$ of the (pseudo) Riemannian metric g with respect to the coordinates $(u^1, \dots, u^m)$ and $(x^1, \dots, x^m)$, respectively, are related as*

$$\tilde{g}_{ab} = \sum_{i,j=1}^{m} \frac{\partial u^i}{\partial x^a} \frac{\partial u^j}{\partial x^b} g_{ij}. \tag{4.6}$$

Moreover, the inverse matrices (g^{ij}) and $(\tilde{g}^{ab})$ of (g_{ij}) and $(\tilde{g}_{ab})$, respectively, satisfy

$$g^{ij} = \sum_{a,b=1}^{m} \frac{\partial u^i}{\partial x^a} \frac{\partial u^j}{\partial x^b} \tilde{g}^{ab}.$$

Proof. By (4.4), we have

$$\begin{aligned} \tilde{g}_{ab} := g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) &= \sum_{i,j=1}^{m} \frac{\partial u^i}{\partial x^a} \frac{\partial u^j}{\partial x^b} g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) \\ &= \sum_{i,j=1}^{m} \frac{\partial u^i}{\partial x^a} \frac{\partial u^j}{\partial x^b} g_{ij} \end{aligned}$$

proving the first assertion. The second assertion follows from (4.3). □

Corollary 4.2. *The Christoffel symbols Γ_{ij}^k as in (3.8) with respect to the coordinate system $(u^1, \dots, u^m)$ and the Christoffel symbols $\widetilde{\Gamma}_{ab}^c$ with respect to $(x^1, \dots, x^m)$ are related as*

$$\widetilde{\Gamma}_{ab}^c = \sum_{k=1}^{m} \frac{\partial x^c}{\partial u^k} \left[\frac{\partial^2 u^k}{\partial x^a \partial x^b} + \sum_{i,j=1}^{m} \frac{\partial u^i}{\partial x^a} \frac{\partial u^j}{\partial x^b} \Gamma_{ij}^k \right].$$

Proof. The definition (3.8) and Lemma 4.1 yields the conclusion through a direct computation. □

The following corollary is essentially a rephrasing of Theorem 3.13. Namely the proof of the corollary gives an alternative proof of Theorem 3.13.

Corollary 4.3. *Let (M, g) be an m-dimensional (pseudo) Riemannian manifold. Then, for each* P*, there exists a coordinate neighborhood $(U; u^1, \dots, u^m)$ of* P *such that the components (g_{ij}) of the metric g satisfy $g_{ij} = \pm\delta_{ij}$ if and only if the Christoffel symbols $\widetilde{\Gamma}_{ab}^c$ of any coordinate system $(x^1, \dots, x^m)$ satisfy* (3.12).

Proof. Let $(V; x^1, \dots, x^m)$ be a coordinate system at P, and denote the Christoffel symbol with respect to (x^a) by $\widetilde{\Gamma}_{ab}^c$. Consider a system of partial differential equations

$$\frac{\partial \mathcal{F}}{\partial x^a} = \mathcal{F} \Omega_a, \tag{4.7}$$

where Ω_a's are matrices defined by (3.13) for $\{\widetilde{\Gamma}_{ab}^c\}$ and $\mathcal{F} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$. Then the integrability condition of (4.7) is equivalent to (3.12) for the Christoffel symbols $\{\widetilde{\Gamma}_{ab}^c\}$ satisfies (3.12).

That is, if $\{\widetilde{\Gamma}_{ab}^c\}$ satisfies (3.12), there exists a solution $\mathcal{F}$ (4.7) with the initial value $\mathcal{F}(\mathrm{P}) = F_0$. In addition, if F_0 is a regular matrix, $\mathcal{F}$ is valued in $\mathrm{GL}(m, \mathbb{R})$.

Moreover, noticing $\widetilde{\Gamma}_{ab}^c = \widetilde{\Gamma}_{ba}^c$, there exists a vector-valued function $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ such that

$$d\boldsymbol{u} = \sum_{a=1}^m \boldsymbol{v}_a dx^a$$

because the right-hand side is a closed one form. Since $\mathcal{F}$ is the Jacobian matrix of $\boldsymbol{x} \mapsto \boldsymbol{u}$, which is valued in $\mathrm{GL}(m, \mathbb{R})$, $\boldsymbol{u} = (u^1, \dots, u^m)$ is a new coordinate system around P.

By Corollary 4.2, the Christoffel symbols with respect to $(u^1, \dots, u^m)$ vanishes identically. This means that g_{ij}'s are constants because of (3.11). Since (g_{ij}) is a constant matrix, a linear transformation of the coordinate system yields the conclusion. □

The Curvature Tensor. Set

(4.8)
$$R_{ijkl} := \sum_{q=1}^m g_{ql} \left(\frac{\partial \Gamma_{ki}^q}{\partial u^j} - \frac{\partial \Gamma_{kj}^q}{\partial u^i} + \sum_{p=1}^m \left(\Gamma_{ik}^p \Gamma_{pj}^q - \Gamma_{kj}^p \Gamma_{pi}^q \right) \right)$$

for $i, j, k, l = 1, \dots, m$, where Γ_{ij}^k's are the Christoffel symbols. Obviously, it holds that

Lemma 4.4. *The (pseudo) Riemannian manifold (M, g) is flat if and only if, for each point $\mathrm{P} \in M$, there exists a coordinate system (u^j) around* P *such that R_{ijkl} $(i, j, k, l = 1, \dots, m)$ vanish identically.*

Here, the condition "there exists a coordinate system" in Lemma 4.4 can be replaced by "for any coordinate systems", because of the following lemma:

Lemma 4.5. *Let $(x^1, \dots, x^m)$ be another coordinate system, and define $\widetilde{R}_{abcd}$ by (4.8) replacing Γ with $\widetilde{\Gamma}$, u with x. Then*

(4.9)
$$\widetilde{R}_{abcd} = \sum_{i,j,k,l=1}^m \frac{\partial u^i}{\partial x^a} \frac{\partial u^j}{\partial x^b} \frac{\partial u^k}{\partial x^c} \frac{\partial u^l}{\partial x^d} R_{ijkl}$$

holds for each a, b, c, $d = 1, \dots, m$.

Proof. By tedious but simple computation, the conclusion follows. □

The relation (4.9) looks similar to (4.6), where the metric g is a notion which is independent of choice of coordinates. In fact, by (4.5) and (4.3), we have

Corollary 4.6. *Let $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ and $\boldsymbol{w} \in T_{\mathrm{P}}M$ and write them by*

$$\boldsymbol{x} = \sum_{i=1}^m x^i \left(\frac{\partial}{\partial u^i} \right)_{\mathrm{P}}, \qquad \boldsymbol{y} = \sum_{j=1}^m y^j \left(\frac{\partial}{\partial u^j} \right)_{\mathrm{P}}$$
$$\boldsymbol{z} = \sum_{k=1}^m z^k \left(\frac{\partial}{\partial u^k} \right)_{\mathrm{P}}, \qquad \boldsymbol{w} = \sum_{l=1}^m w^l \left(\frac{\partial}{\partial u^l} \right)_{\mathrm{P}}.$$

Then

$$R(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}) := \sum_{i,j,k,l=1}^{m} x^i y^j z^k w^l R_{ijkl} \tag{4.10}$$

does not depend on choice of coordinates.

Thus, we can define a 4-linear map

$$R\colon T_{\mathrm{P}}M \times T_{\mathrm{P}}M \times T_{\mathrm{P}}M \times T_{\mathrm{P}}M \longrightarrow \mathbb{R},$$

and

$$R\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M),$$

where $C^\infty(M)$ is the commutative ring consists of C^∞-functions on M, and $\mathfrak{X}(M)$ is the $C^\infty(M)$-module consists of smooth vector fields on M. In fact, for X, Y, Z, W, we define

$$R(X,Y,Z,W)\colon M \ni \mathrm{P} \mapsto R(X_{\mathrm{P}},Y_{\mathrm{P}},Z_{\mathrm{P}},W_{\mathrm{P}}) \in \mathbb{R}.$$

Then R is $C^\infty(M)$-linear in each entry, namely, for X, Y, Z ,$W \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$,

$$\begin{aligned} R(fX,Y,Z,W) &= R(X,fY,Z,W) = R(X,Y,fZ,W) \\ &= R(X,Y,Z,fW) = fR(X,Y,Z,W) \end{aligned} \tag{4.11}$$

holds. We call this R the *curvature tensor* of (M,g). Theorem 3.13 can be restated as following “coordinate free” form.

Corollary 4.7. *The Riemannian manifold is flat if and only if its curvature tensor vanishes identically.*

Covariant Derivatives. To define the curvature tensor in the coordinate-free form, we introduce the notion of *covariant derivatives* of vector fields.

For a vector field Y and tangent vector $\boldsymbol{v} \in T_{\mathrm{P}}M$, we define

$$\nabla_{\boldsymbol{v}} Y := \sum_{j=1}^{m} \left[\sum_{k=1}^{m} v^k \left(\frac{\partial Y^j}{\partial u^k} + \sum_{l=1}^{m} \Gamma^j_{lk} Y^l \right) \right] \left(\frac{\partial}{\partial u^j} \right)_{\mathrm{P}}, \tag{4.12}$$

where $Y = \sum_{i=1}^m Y^i(\partial/\partial u^i)$ and $\boldsymbol{v} = \sum_{i=1}^m v^i(\partial/\partial u^i)_{\mathrm{P}}$, and Γ^k_{ij}’s are the Christoffel symbols defined in (3.8).

Proposition 4.8. *Assume M is a (non-degenerate) submanifold of the (pseudo) Euclidean space $\mathbb{R}^{n+1}_s$, and take a vector field X on M defined on a neighborhood of $\mathrm{P} \in M$. Then*

$$\nabla_{\boldsymbol{v}} X = [D_{\boldsymbol{v}} X]^{\mathrm{T}}$$

holds, where $D_{\boldsymbol{v}}X$ is a directional derivative of $\mathbb{R}^{n+1}_s$-valued function with respect to $\boldsymbol{v}$, and $[*]^{\mathrm{T}}$ denotes the tangential component of it, as in (3.1).

Proof. Let $f = f(u^1,\dots,u^m)$ be a parametrization of M with respect to the local coordinate system (u^j) and we let

$$\left[\frac{\partial^2 f}{\partial u^i \partial u^j} \right]^{\mathrm{T}} = \sum_{k=1}^{m} G^k_{ij} \frac{\partial f}{\partial u^k}.$$

Then by (3.10) and (3.4), we have

$$\sum_{k=1}^{m} g_{kl} G^k_{ij} = \sum_{k=1}^{m} g_{kl} \Gamma^k_{ij}.$$

Since (g_{ij}) is a regular matrix, we have $G^k_{ij} = \Gamma^k_{ij}$. In other words,

$$\left[\frac{\partial^2 f}{\partial u^i \partial u^j}\right]^{\mathrm{T}} = \sum_{k=1}^{m} \Gamma^k_{ij} \frac{\partial f}{\partial u^k} \tag{4.13}$$

holds. Thus, identifying $\partial/\partial u^j$ with $\partial f/\partial u^j$, we have

$$\left[D_{\partial/\partial u^i} \frac{\partial f}{\partial u^j}\right]^{\mathrm{T}} = \left[\frac{\partial^2 f}{\partial u^i \partial u^k}\right]^{\mathrm{T}} = \sum_{k=1}^{m} \Gamma^k_{ij} \frac{\partial f}{\partial u^k} = \nabla_{\partial/\partial u^i} \frac{\partial f}{\partial u^j}.$$

Applying this, the conclusion follows. □

Using covariant derivative, we obtain the bilinear

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto \nabla_X Y \in \mathfrak{X}(M), \tag{4.14}$$

which is also called the *covariant derivative*, alternatively, the *Riemannian connection* or the *Levi-Civita connection*.

Proposition 4.9. *For each X, $Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$,*

$$\nabla_{fX} Y = f \nabla_X Y, \tag{4.15}$$

$$\nabla_X fY = (Xf)Y + f\nabla_X Y, \tag{4.16}$$

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{4.17}$$

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{4.18}$$

where $[\ ,\ ]$ denotes the Lie-bracket for vector fields.

Proof. The first two assertions are direct conclusion of the definition of ∇. The third assertion follows because $\Gamma^k_{ij} = \Gamma^k_{ji}$ (cf. (3.9)) The last assertion can be proved by

$$\frac{\partial g_{ij}}{\partial u^l} = \sum_{k=1}^{m} \left(g_{kj} \Gamma^k_{il} + g_{ik} \Gamma^k_{jl} \right),$$

as seen in (3.11). □

Proposition 4.10. *For X, Y, Z and $W \in \mathfrak{X}(M)$, it holds that*

$$R(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W), \tag{4.19}$$

where R is the curvature tensor as in (5.7).

Proof. Denote the right-hand side of (4.19) by $S(X, Y, Z, W)$. Then by Proposition 4.9, it holds that

$$\begin{aligned} S(fX, Y, Z, W) = S(X, fY, Z, W) = S(X, Y, fZ, W) \\ S(X, Y, Z, fW) = fS(X, Y, Z, W). \end{aligned} \tag{4.19a}$$

Then by (4.19a) it is sufficient to show the conclusion for

$$X = \frac{\partial}{\partial u^i}, \quad Y = \frac{\partial}{\partial u^j}, \quad Z = \frac{\partial}{\partial u^k}, \quad W = \frac{\partial}{\partial u^l}. \tag{4.20}$$

In fact,

$$
\begin{aligned}
\nabla_{\partial/\partial u^j}\frac{\partial}{\partial u^i} &= \sum_{l=1}^{m}\Gamma_{ij}^{l}\frac{\partial}{\partial u^k},\\
\nabla_{\partial/\partial u^k}\nabla_{\partial/\partial u^j}\frac{\partial}{\partial u^i} &= \sum_{l=1}^{m}\left[\frac{\partial\Gamma_{ij}^{l}}{\partial u^k}\frac{\partial}{\partial u^l}+\Gamma_{ij}^{l}\nabla_{\partial/\partial u^k}\frac{\partial}{\partial u^l}\right]\\
&= \left[\frac{\partial\Gamma_{ij}^{l}}{\partial u^k}+\Gamma_{ij}^{p}\Gamma_{pk}^{l}\right]\frac{\partial}{\partial u^l},
\end{aligned}
$$

and

$$
\left[\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j}\right]=0
$$

yield the conclusion. □

Exercises

4-1 Prove Corollary 4.2.

4-2 We consider a Riemannian metric g on a domain $U \subset \mathbb{R}^2$ with

$$
g_{11}=g_{22}=e^{2\sigma}, \qquad g_{12}=g_{21}=0,
$$

with respect to the canonical coordinate system (u^1,u^2), where σ is a smooth function on U.

(1) Show that (U,g) is flat if and only if σ is a harmonic function, that is, it satisfies

$$
\frac{\partial^2\sigma}{(\partial u^1)^2}+\frac{\partial^2\sigma}{(\partial u^2)^2}=0.
$$

(2) Compute R_{ijkl} for

$$
e^{2\sigma}=\frac{4}{(1+k(u^2+v^2))^2},
$$

where k is a constant and $(u,v)=(u^1,u^2)$ is the canonical coordinate system on $\mathbb{R}^2$.