# 3 Flatness.

**Riemannian manifolds.** A *Riemannian manifold* (resp. *pseudo Riemannian manifold*) is a pair $(M, g)$ of a $C^\infty$-manifold $M$ and a *Riemannian metric* (resp. *pseudo Riemannian metric*) on $M$, that is, $g$ is a collection $\{g_{\mathrm{P}}\,;\,\mathrm{P} \in M\}$ of positive definite (resp. non-degenerate) inner products on $T_{\mathrm{P}}M$ such that, for each pair of $C^\infty$-vector fields $(X, Y)$ on $M$, the map

$$M \ni \mathrm{P} \longmapsto g_{\mathrm{P}}(X_{\mathrm{P}}, Y_{\mathrm{P}}) \in \mathbb{R}$$

is a $C^\infty$-function, where $X_{\mathrm{P}}$ and $Y_{\mathrm{P}}$ are values of $X$ and $Y$ at P, respectively.

**Example 3.1** ((Pseudo) Euclidean spaces.)**.** Let $\mathbb{R}^n_s$ be a pseudo Euclidean vector space with inner product $\langle\ ,\ \rangle$ of signature $(n-s, s)$. Identifying the tangent space $T_{\mathrm{P}}\mathbb{R}^n_s$ of $\mathbb{R}^n_s$ itself by translations, $\langle\ ,\ \rangle$ gives a pseudo Riemannian metric of the manifold $\mathbb{R}^n_s$. Such a pseudo Riemannian manifold is called the *pseudo Euclidean space* of signature $(n-s, s)$. In particular, $\mathbb{R}^n := \mathbb{R}^n_0$, which is a Riemannian manifold, is called the Euclidean space.

**Example 3.2.** Let $M$ be a submanifold of the Euclidean space $\mathbb{R}^n$, that is, $M$ is a subset of $\mathbb{R}^n$ and has a structure of $C^\infty$-manifold such that the inclusion map $\iota\colon M \to \mathbb{R}^n$ is an immersion. Then $T_{\mathrm{P}}M$ is considered as a linear subspace of $\mathbb{R}^n (= T_{\mathrm{P}}\mathbb{R}^n)$, and then the restriction of the inner product $\langle\ ,\ \rangle$ of $\mathbb{R}^n$ gives a Riemannian metric on $M$. Such a Riemannian metric is called the *induced metric*.

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**Example 3.3** (Spheres)**.** For positive real number $k$, a subset $S^n(k) := \{\boldsymbol{x} \in \mathbb{R}^{n+1}\,;\,\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1/k\}$ is an $n$-dimensional $C^\infty$-submanifold (hypersurface)[5] of the Euclidean space $\mathbb{R}^{n+1}$, called the $n$-dimensional *sphere* of curvature[6] $k$.

**Example 3.4.** A linear subspace $L$ of $\mathbb{R}^n_s$ is said to be *non-degenerate* if the restriction of the inner product $\langle\ ,\ \rangle$ of $\mathbb{R}^n_s$ to $L$ is non-degenerate. A submanifold $M \subset \mathbb{R}^n_s$ is said to be *non-degenerate* if $T_{\mathrm{P}}M$ is a non-degenerate subspace of $\mathbb{R}^n_s$ for each $\mathrm{P} \in M$. In this case, the restriction of $\langle\ ,\ \rangle$ on $T_{\mathrm{P}}M$ is a (non-degenerate) inner product of $T_{\mathrm{P}}M$.

**Example 3.5** (Hyperbolic spaces.)**.** For positive real number $k$, a subset

$$H^n(-k) := \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1\,;\,\langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1/k, x^0 > 0\}$$

is a connected $C^\infty$-hypersurface of the Lorentz-Minkowski space $\mathbb{R}^{n+1}_1$, where $\langle\ ,\ \rangle$ is the inner product of signature $(-,+,\dots,+)$, and $\boldsymbol{x} = {}^t(x^0, x^1, \dots, x^n)$. The tangent space $T_{\boldsymbol{x}}H^n(-k) = \{\boldsymbol{v} \in \mathbb{R}^{n+1}_1\,;\,\langle \boldsymbol{v}, \boldsymbol{x}\rangle = 0\} = \boldsymbol{x}^\perp$ is non-degenerate subspace in $\mathbb{R}^{n+1}_1$ and the restriction of $\langle\ ,\ \rangle$ to $T_{\boldsymbol{x}}H^n(-k)$ is positive definite. Thus, we obtain a Riemannian manifold $H^n(-k)$, which is called the *hyperbolic space* of curvature $-k$.

**Geodesics.** Let $M \subset \mathbb{R}^{n+1}_s$ be a non-degenerate submanifold of dimension $m$. By non-degeneracy, the orthogonal decompo-

[5] An $n$-dimensional submanifold of $(n+1)$-dimensional manifold (i.e., a submanifold of codimension one) is called a *hypersurface*.

[6] The word *curvature* is undefined at the moment.

sition

$$\mathbb{R}^{n+1}_s = T_{\mathrm{P}}\mathbb{R}^{n+1}_s = T_{\mathrm{P}}M \oplus N_{\mathrm{P}}, \bigl(N_{\mathrm{P}} := (T_{\mathrm{P}}M)^{\perp}\bigr) \tag{3.1}$$

holds for each $\mathrm{P} \in M$. Take a curve $\gamma$ on $M$, that is, $\gamma$ is a $C^\infty$-map

$$\gamma\colon J \ni t \longmapsto \gamma(t) \in M \subset \mathbb{R}^{n+1}_s$$

where $J \subset \mathbb{R}$ is an interval. From now on, by a word *smooth*, we mean "of class $C^\infty$".

**Definition 3.6.** Let $\gamma\colon J \to M \subset \mathbb{R}^{n+1}_s$ be a smooth curve on $M$. A *smooth vector field on $M$ along $\gamma$* is a map

$$X\colon J \ni t \longmapsto X(t) \in T_{\gamma(t)}M \subset \mathbb{R}^{n+1}_s$$

which is of class $C^\infty$ as a map from $J$ to $\mathbb{R}^{n+1}_s$.

**Example 3.7.** Let $\gamma\colon J \to M \subset \mathbb{R}^{n+1}_s$ be a smooth curve. Then

$$\dot\gamma\colon J \ni t \longmapsto \dot\gamma(t) = \frac{d\gamma}{dt}(t) \in T_{\gamma(t)}M$$

is a smooth vector field along $\gamma$, called the *velocity* vector field of the curve $\gamma$.

**Definition 3.8.** Let $X$ be a smooth vector field along a smooth curve $\gamma$ on $M$. Then the vector field

$$\frac{\nabla}{dt}X(t) := \nabla_{\dot\gamma(t)}X(t) := \Bigl[\dot X(t)\Bigr]^{\mathrm{T}} \in T_{\gamma(t)}M$$

of $M$ along $\gamma$ is called the *covariant derivative* of $X$ along $\gamma$, where $[*]^{\mathrm{T}}$ denotes the tangential component as in (3.1).



**Definition 3.9.** The covariant derivative

$$\frac{\nabla}{dt}\dot\gamma(t) = \nabla_{\dot\gamma(t)}\dot\gamma(t) := [\ddot\gamma(t)]^{\mathrm{T}} \in T_{\gamma(t)}M \tag{3.2}$$

of $\dot\gamma$ along $\gamma$ is called the *acceleration* of the curve $\gamma$.

**Definition 3.10.** A curve $\gamma$ on a non-degenerate submanifold $M \subset \mathbb{R}^{n+1}_s$ is called a *geodesic* if $\nabla_{\dot\gamma}\dot\gamma$ vanishes identically.

**Local and intrinsic expressions.** Let $M \subset \mathbb{R}^{n+1}_s$ be a non-degenerate submanifold and take a local coordinate neighborhood $(U; u^1, \dots, u^m)$ of $M$, where $m = \dim M$. Then the inclusion map $\iota\colon M \to \mathbb{R}^{n+1}_s$ induces an immersion

$$f\colon U \ni (u^1, \dots, u^m) \longmapsto f(u^1, \dots, u^m) \in M \subset \mathbb{R}^{n+1}_s, \tag{3.3}$$

here we identify the coordinate neighborhood $U \subset M$ with a region of $\mathbb{R}^m$. We call such an $f$ a (local) *parametrization* of $M$. Under this parametrization, the canonical basis $\{(\partial/\partial u^j)_{\mathrm{P}}\}$ of $T_{\mathrm{P}}M$ (in the abstract way) is identified with

$$\left\{\frac{\partial f}{\partial u^1}(\mathrm{P}), \dots, \frac{\partial f}{\partial u^m}(\mathrm{P})\right\} \subset T_{\mathrm{P}}M \subset \mathbb{R}^{n+1}_s.$$

We set, for $i, j = 1, \dots, m$,

$$g_{ij} := g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = \left\langle \frac{\partial f}{\partial u^i}, \frac{\partial f}{\partial u^j}\right\rangle (= g_{ji}) \tag{3.4}$$

which is a component of the induced metric $g := \langle\ ,\ \rangle|_{T_{\mathrm{P}}M}$ with respect to the canonical basis $\{\partial/\partial u^j\}$. Since the induced metric is non-degenerate, the $m \times m$-matrix $(g_{ij})$ is a regular matrix at each point $\mathrm{P} \in M$. In particular, when the induced metric is positive definite, $(g_{ij})$ is positive definite. We denote by $(g^{ij})$ the inverse matrix of $(g_{ij})$:

$$\sum_{k=1}^{m} g_{ik} g^{kj} = \delta_i^j = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{3.5}$$

Then, as we have seen in Section 5 of "Advanced Topics in Geometry A1, 2019" (the previous quarter), we have

**Lemma 3.11.** *Let $\gamma$ is a curve in $U \subset M$ and express*

$$\gamma(t) = f\bigl(u^1(t), \dots, u^m(t)\bigr),$$

*where $f\colon U \to M$ is a local parametrization of $M$ as in* (3.3).

$$\dot\gamma = \sum_{j=1}^{m} \frac{du^j}{dt} \frac{\partial f}{\partial u^j} \tag{3.6}$$

$$\nabla_{\dot\gamma}\dot\gamma = \sum_{j=1}^{m} \left( \frac{d^2 u^j}{dt^2} + \sum_{k,l=1}^{m} \Gamma^j_{kl} \frac{du^k}{dt} \frac{du^l}{dt} \right) \frac{\partial f}{\partial u^j} \tag{3.7}$$

*hold, where*

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{m} g^{kl} \left( \frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{lj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} \right). \tag{3.8}$$

The functions $\Gamma^k_{ij}$ of (3.8) are called the *Christoffel symbols* with respect to the local coordinate system $(u^1, \dots, u^m)$.



**Christoffel symbols.** By definition (3.8), the Christoffel symbols $\Gamma^k_{ij}$ are functions defined on the coordinate neighborhood $U$ which are determined only by the coefficients $(g_{ij})$ of the (pseudo) Riemannian metric. That is, the definition of $\Gamma^k_{ij}$ does not require the knowledge of $\gamma$.

**Proposition 3.12.** *Let $M \subset \mathbb{R}^{n+1}_s$ be a non-degenerate submanifold with induced metric $\langle\ ,\ \rangle$, and take a local coordinate system $(U; u^1, \dots, u^m)$ of $M$. We write parametrization of $M$ with respect to $(u^j)$ as* (3.3). *Then the Christoffel symbols $\Gamma^k_{ij}$ with respect to $(u^j)$ satisfy*

$$\Gamma^k_{ij} = \Gamma^k_{ji} \tag{3.9}$$

$$\left\langle \frac{\partial^2 f}{\partial u^i \partial u^j}, \frac{\partial f}{\partial u^l} \right\rangle = \sum_{k=1}^{m} g_{lk} \Gamma^k_{ij}, \tag{3.10}$$

$$\frac{\partial g_{ij}}{\partial u^l} = \sum_{k=1}^{m} \left( g_{kj} \Gamma^k_{il} + g_{ik} \Gamma^k_{jl} \right), \tag{3.11}$$

*where $g_{ij}$'s are the components of the induced metric defined in* (3.4).

*Proof.* The first equality (3.9) is obvious from the definition

(3.8) and the symmetricity of $(g_{ij})$. Next, we prove (3.10). Since

$$
\begin{aligned}
\left\langle \frac{\partial^2 f}{\partial u^i \partial u^j}, \frac{\partial f}{\partial u^l} \right\rangle &= \frac{\partial}{\partial u^i} \left\langle \frac{\partial f}{\partial u^j}, \frac{\partial f}{\partial u^l} \right\rangle - \left\langle \frac{\partial f}{\partial u^j}, \frac{\partial^2 f}{\partial u^i \partial u^l} \right\rangle \\
&= \frac{\partial g_{jl}}{\partial u^i} - \left\langle \frac{\partial f}{\partial u^j}, \frac{\partial^2 f}{\partial u^l \partial u^i} \right\rangle \\
&= \frac{\partial g_{lj}}{\partial u^i} - \frac{\partial}{\partial u^l} \left\langle \frac{\partial f}{\partial u^j}, \frac{\partial f}{\partial u^i} \right\rangle + \left\langle \frac{\partial^2 f}{\partial u^l \partial u^j}, \frac{\partial f}{\partial u^i} \right\rangle \\
&= \frac{\partial g_{lj}}{\partial u^i} - \frac{\partial g_{ji}}{\partial u^l} + \left\langle \frac{\partial^2 f}{\partial u^j \partial u^l}, \frac{\partial f}{\partial u^i} \right\rangle \\
&= \frac{\partial g_{lj}}{\partial u^i} - \frac{\partial g_{ji}}{\partial u^l} + \frac{\partial}{\partial u^j} \left\langle \frac{\partial f}{\partial u^l}, \frac{\partial f}{\partial u^i} \right\rangle - \left\langle \frac{\partial f}{\partial u^l}, \frac{\partial^2 f}{\partial u^j \partial u^i} \right\rangle \\
&= \frac{\partial g_{lj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} + \frac{\partial g_{li}}{\partial u^j} - \left\langle \frac{\partial^2 f}{\partial u^i \partial u^j}, \frac{\partial f}{\partial u^l} \right\rangle,
\end{aligned}
$$

we have

$$
\begin{aligned}
\left\langle \frac{\partial^2 f}{\partial u^i \partial u^j}, \frac{\partial f}{\partial u^l} \right\rangle &= \frac{1}{2} \left( \frac{\partial g_{lj}}{\partial u^i} + \frac{\partial g_{li}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l} \right) \\
&= \frac{1}{2} \sum_{p=1}^{m} \delta_l^p \left( \frac{\partial g_{pj}}{\partial u^i} + \frac{\partial g_{pi}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^p} \right) \\
&= \frac{1}{2} \sum_{k,p=1}^{m} g_{lp} g^{pk} \left( \frac{\partial g_{pj}}{\partial u^i} + \frac{\partial g_{pi}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^p} \right) = \sum_{k=1}^{m} g_{kl} \Gamma_{ij}^k.
\end{aligned}
$$



Using this, we have

$$
\begin{aligned}
\frac{\partial g_{ij}}{\partial u^l} &= \frac{\partial}{\partial u^l} \left\langle \frac{\partial f}{\partial u^i}, \frac{\partial f}{\partial u^j} \right\rangle = \left\langle \frac{\partial^2 f}{\partial u^l \partial u^i}, \frac{\partial f}{\partial u^j} \right\rangle + \left\langle \frac{\partial f}{\partial u^i}, \frac{\partial^2 f}{\partial u^l \partial u^j} \right\rangle \\
&= \sum_{k=1}^{m} g_{kj} \Gamma_{li}^k + \sum_{k=1}^{m} g_{ik} \Gamma_{lj}^k,
\end{aligned}
$$

proving (3.11). □

**Flatness.** We shall prove the following:

**Theorem 3.13.** *Let $(M, g)$ be a Riemannian manifold of dimension $m$ (resp. pseudo Riemannian manifold of signature $(m-s, s)$), and $(U; u^1, \dots, u^m)$ a local coordinate system. Assume there exists an immersion $f\colon U \to \mathbb{R}^m$ (resp. $\mathbb{R}^m_s$) into the Euclidean space (resp. the pseudo Euclidean space) with the same dimension $m$ as $M$ whose induced metric is $g$. Then the Christoffel symbols $\Gamma_{ij}^k$ with respect to the coordinate system $(u^j)$ satisfy*

$$
\frac{\partial \Gamma_{ij}^l}{\partial u^k} - \frac{\partial \Gamma_{ik}^l}{\partial u^j} + \sum_{p=1}^{m} \left( \Gamma_{ij}^p \Gamma_{pk}^l - \Gamma_{ik}^p \Gamma_{pj}^l \right) = 0 \tag{3.12}
$$

*holds for $i, j, k, l = 1, \dots, m$. Conversely, when $U$ is simply connected and (3.12) holds, there exists an immersion $f\colon U \to \mathbb{R}^m$ (resp. $\mathbb{R}^m_s$) such that the induced metric by $f$ coincides with the metric $g$.*

**Lemma 3.14.** *Let $f\colon U \to \mathbb{R}^m$ (resp. $\mathbb{R}^m_s$) be an immersion of a domain $(U; u^1, \dots, u^m) \subset \mathbb{R}^m$, and set*

$$\mathcal{F} := \left(\frac{\partial f}{\partial u^1}, \dots, \frac{\partial f}{\partial u^m}\right) : U \to \mathrm{M}_m(\mathbb{R}).$$

*Then $\mathcal{F}$ satisfies*

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j, \quad \Omega_j := \begin{pmatrix} \Gamma^1_{1j} & \dots & \Gamma^1_{mj} \\ \vdots & \ddots & \vdots \\ \Gamma^m_{1j} & \dots & \Gamma^m_{mj} \end{pmatrix} \tag{3.13}$$

*for $j = 1, \dots, m$, where $\Gamma^k_{ij}$'s are the Christoffel symbols of the induced metric with respect to the coordinate system $(u^j)$. Moreover, the Christoffel symbols satisfy* (3.12).

*Proof.* Problem 3-1. □

**Lemma 3.15.** *Let $A \in \mathrm{M}_m(\mathbb{R})$ be a symmetric matrix such that the quadratic form $\boldsymbol{x} \mapsto {}^t\boldsymbol{x}A\boldsymbol{x}$ has a signature $(s, m-s)$. Then there exists a regular matrix $P$ such that*

$${}^tPJ_{s,m-s}P = A, \qquad J_{s,m-s} = \begin{pmatrix} -\mathrm{id}_s & O \\ O & \mathrm{id}_{m-s} \end{pmatrix},$$

*where $\mathrm{id}_k$ is the $k \times k$ identity matrix and $O$'s are zero matrices.*

*Proof.* By the assumptions, $A$ has $(m-s)$ positive eigenvalues and $s$ negative eigenvalues, and $A$ can be diagonalized by an orthogonal matrix $Q$:

$$A = {}^tQ \begin{pmatrix} \Lambda_- & O \\ O & \Lambda_+ \end{pmatrix} Q,$$



where $\Lambda_- := \mathrm{diag}(-a_1^2, \dots, -a_s^2)$, $\Lambda_+ := \mathrm{diag}(a_{s+1}^2, \dots, a_m^2)$ and $a_j$'s $(j = 1, \dots, m)$ are non-zero real numbers. Let $D := \mathrm{diag}(a_1, \dots, a_m)$ and $P := DQ$, we have the conclusion. □

**Proof of Theorem 3.13.** The first assertion has been proved in Lemma 3.14. We assume (3.12) holds for each $i$, $j$, $k$, $l = 1, \dots, m$, and fix $\mathrm{P}_0 \in U$. We let $A = (g_{ij}(\mathrm{P}_0))$, which is a symmetric matrix such that the corresponding quadratic form is of signature $(m-s, s)$. Then there exists a regular matrix $P$ as in Lemma 3.15. Then by Theorem 2.5, there exists $\mathcal{F}\colon U \to \mathrm{M}_m(\mathbb{R})$ satisfying (3.13) with initial condition $\mathcal{F}(\mathrm{P}_0) = P$. We set

$$\omega := \sum_{j=1}^m \boldsymbol{f}_j\, du^j, \qquad \text{where} \quad \mathcal{F} = (\boldsymbol{f}_1, \dots, \boldsymbol{f}_m).$$

Then by (3.9), we know that $\omega$ is a vector-valued closed one form. Hence by Poincaré's Lemma (Theorem 2.6), there exists a $C^\infty$-function $f\colon M \to \mathbb{R}^m_s$ such that $df = \omega$, that is,

$$\frac{\partial f}{\partial u^j} = \boldsymbol{f}_j \qquad (j = 1, \dots, m).$$

We shall prove that this $f$ is the desired immersion, that is, our goal is to prove

$$g_{ij} = \langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle \qquad (i, j = 1, \dots, m).$$

To do it, we set

$$\kappa_{ij} := g_{ij} - \langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle \qquad (i, j = 1, \dots, m).$$

So, by a choice of the initial condition, we have

$$\kappa_{ij}(\mathrm{P}_0) = 0 \qquad (i, j = 1, \dots, m). \tag{3.14}$$

Then, by (3.13) and (3.11), it holds that

$$\frac{\partial \kappa_{ij}}{\partial u^l} = \sum_{k=1}^{m} \left(\kappa_{ik} \mathit{\Gamma}_{jl}^k + \kappa_{kj} \mathit{\Gamma}_{il}^k\right) \qquad (i, j, l = 1, \dots, m). \tag{3.15}$$

Let $\mathrm{P} \in U$ and take a path $\gamma(t)$ $(0 \leqq t \leqq 1)$ in $U$ satisfying $\gamma(0) = \mathrm{P}_0$ and $\gamma(1) = \mathrm{P}$. Then the functions $\tilde{\kappa}_{ij}(t)$ satisfy a system of ordinary differential equations

$$\frac{d\tilde{\kappa}_{ij}}{dt} = \sum_{l=1}^{m} \sum_{k=1}^{m} \left(\tilde{\kappa}_{ik} \mathit{\Gamma}_{jl}^k \circ \gamma + \tilde{\kappa}_{kj} \mathit{\Gamma}_{il}^k \circ \gamma\right) \frac{du^l}{dt} \quad (i, j = 1, \dots, m),$$

where $\gamma(t) = (u^1(t), \dots, u^m(t))$. Since $\tilde{\kappa}_{ij}(t) = 0$ $(i, j = 1, \dots, m)$ satisfy the equation with initial condition (3.14), uniqueness theorem implies that $\tilde{\kappa}_{ij}(1) = \kappa_{ij}(\mathrm{P}) = 0$, proving the theorem.

*Remark* 3.16. As we see in the following section, the condition (3.12) does not depend on choice of local coordinate systems. We say a (pseudo) Riemmanian manifold $(M, g)$ to be *flat* if (3.12) holds on $M$.



***Exercises***

**3-1** Show Lemma 3.14.

**3-2** Let $M := \mathbb{R}_+ \times \mathbb{R} = \{(u^1, u^2)\,;\, u^1 > 0\}$, and consider a Riemannian metric $g$ on $M$ whose components are

$$g_{11} = 1, \qquad g_{12} = 0, \qquad g_{22} = \left\{\varphi(u^1)\right\}^2,$$

where $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function.

- Find a function $\varphi$ satisfying (3.12) and $\lim_{t \to 0+} \varphi(t) = 0$.
- Under the situation above, find $f\colon U \to \mathbb{R}^2$ on an appropriate domain $U$ on $M$ such that the induced metric coincides with $g$.