2 Integrability Conditions

Let $U \subset \mathbb{R}^m$ be a domain of $(\mathbb{R}^m; u^1, \dots, u^m)$ and consider m-tuple of $n \times n$-matrix valued C^∞-maps

$$(2.1) \qquad \Omega_j \colon \mathbb{R}^m \supset U \longrightarrow \mathrm{M}_n(\mathbb{R}) \qquad (j = 1, \dots, m).$$

In this section, we consider an initial value problem of a system of linear partial differential equations

$$(2.2) \qquad \frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0,$$

where $\mathrm{P}_0 = (u_0^1, \dots, u_0^m) \in U$ is a fixed point, X is an $n \times n$-matrix valued unknown, and $X_0 \in \mathrm{M}_n(\mathbb{R})$. The chain rule yields the following:

Lemma 2.1. *Let $X \colon U \to \mathrm{M}_n(\mathbb{R})$ be a C^∞-map satisfying* (2.2)*. Then for each smooth path $\gamma \colon I \to U$ defined on an interval $I \subset \mathbb{R}$, $\hat{X} := X \circ \gamma : I \to \mathrm{M}_n(\mathbb{R})$ satisfies the ordinary differential equation*

$$(2.3) \quad \frac{d\hat{X}}{dt}(t) = \hat{X}(t)\Omega_\gamma(t) \qquad \left(\Omega_{\gamma(t)} := \sum_{j=1}^n \Omega_j \circ \gamma(t) \frac{du^j}{dt}(t)\right)$$

on I, where $\gamma(t) = \bigl(u^1(t), \dots, u^m(t)\bigr)$.

Proposition 2.2. *If a C^∞-map $X \colon U \to \mathrm{M}_n(\mathbb{R})$ defined on a domain $U \subset \mathbb{R}^m$ satisfies* (2.2) *with $X_0 \in \mathrm{GL}(n, \mathbb{R})$, then $X(\mathrm{P}) \in \mathrm{GL}(n, \mathbb{R})$ for all $\mathrm{P} \in U$. In addition, if Ω_j (j=1,...,m) are skew-symmetric and $X_0 \in \mathrm{SO}(n)$, then $X(\mathrm{P}) \in \mathrm{SO}(n)$ holds for all $\mathrm{P} \in U$.*

Proof. Since U is connected, there exists a continuous path $\gamma_0 \colon [0,1] \to U$ such that $\gamma_0(0) = \mathrm{P}_0$ and $\gamma_0(1) = \mathrm{P}$. By Whitney's approximation theorem (cf. Theorem 10.16 in [2-3]), there exists a smooth path $\gamma \colon [0,1] \to U$ joining P_0 and P approximating γ_0. Since $\hat{X} := X \circ \gamma$ satisfies (2.3) with $\hat{X}(0) = X_0$, Proposition 1.3 yields that $\det \hat{X}(1) \neq 0$ whenever $\det X_0 \neq 0$. The latter half follows from Proposition 1.4. □

Proposition 2.3. *If a matrix-valued C^∞ function $X \colon U \to \mathrm{GL}(n, \mathbb{R})$ satisfies* (2.2)*, it holds that*

$$(2.4) \qquad \frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j$$

for each (j, k) with $1 \leqq j < k \leqq n$.

Proof. Differentiating (2.2) by u^k, we have

$$\frac{\partial^2 X}{\partial u^k \partial u^j} = \frac{\partial X}{\partial u^k}\Omega_j + X\frac{\partial \Omega_j}{\partial u^k} = X\left(\frac{\partial \Omega_j}{\partial u^k} + \Omega_k \Omega_j\right).$$

On the other hand, switching the roles of j and k, we get

$$\frac{\partial^2 X}{\partial u^j \partial u^k} = X\left(\frac{\partial \Omega_k}{\partial u^j} + \Omega_j \Omega_k\right).$$

Since X is of class C^∞, the left-hand sides of these equalities coincide, and so are the right-hand sides. Since $X \in \mathrm{GL}(n, \mathbb{R})$, the conclusion follows. □

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The equality (2.4) is called the *integrability condition* or *compatibility condition* of (2.2).

Lemma 2.4. *Let $\Omega_j\colon U \to \mathrm{M}_n(\mathbb{R})$ $(j=1,\dots,m)$ be C^∞-map defined on a domain $U \subset \mathbb{R}^m$ which satisfy (2.4). Then for each smooth map*

$$\sigma\colon D \ni (t,w) \longmapsto \sigma(t,w) = (u^1(t,w),\dots,u^m(t,w)) \in U$$

defined on a domain $D \subset \mathbb{R}^2$, it holds that

$$\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} - TW + WT = 0, \tag{2.5}$$

where

$$T := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial t}, \quad W := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial w} \quad (\widetilde{\Omega}_j := \Omega_j \circ \sigma). \tag{2.6}$$

Proof. By the chain rule, we have

$$\begin{aligned}
\frac{\partial T}{\partial w} &= \sum_{j,k=1}^m \frac{\partial \Omega_j}{\partial u^k}\frac{\partial u^k}{\partial w}\frac{\partial u^j}{\partial t} + \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial^2 u^j}{\partial w \partial t},\\
\frac{\partial W}{\partial t} &= \sum_{j,k=1}^m \frac{\partial \Omega_j}{\partial u^k}\frac{\partial u^k}{\partial t}\frac{\partial u^j}{\partial w} + \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial^2 u^j}{\partial t \partial w}\\
&= \sum_{j,k=1}^m \frac{\partial \Omega_k}{\partial u^j}\frac{\partial u^j}{\partial t}\frac{\partial u^k}{\partial w} + \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial^2 u^j}{\partial t \partial w}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} &= \sum_{j,k=1}^m \left(\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j}\right)\frac{\partial u^k}{\partial w}\frac{\partial u^j}{\partial t}\\
&= \sum_{j,k=1}^m (\Omega_j\Omega_k - \Omega_k\Omega_j)\frac{\partial u^k}{\partial w}\frac{\partial u^j}{\partial t}\\
&= \left(\sum_{j=1}^m \Omega_j \frac{\partial u^j}{\partial t}\right)\left(\sum_{k=1}^m \Omega_k \frac{\partial u^k}{\partial w}\right) - \left(\sum_{k=1}^m \Omega_k \frac{\partial u^k}{\partial w}\right)\left(\sum_{j=1}^m \Omega_j \frac{\partial u^j}{\partial t}\right)\\
&= TW - WT. \qquad \square
\end{aligned}$$

Integrability of linear systems. In this section, we shall prove the following

Theorem 2.5. *Let $\Omega_j\colon U \to \mathrm{M}_n(\mathbb{R})$ $(j=1,\dots,m)$ bed C^∞-functions defined on a simply connected domain $U \subset \mathbb{R}^m$ satisfying (2.4). Then for each $\mathrm{P}_0 \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique $n\times n$-matrix valued function $X\colon U \to \mathrm{M}_n(\mathbb{R})$ (2.2). Moreover,*

- *if $X_0 \in \mathrm{GL}(n,\mathbb{R})$, $X(\mathrm{P}) \in \mathrm{GL}(n,\mathbb{R})$ holds on U,*
- *if $X_0 \in \mathrm{SO}(n)$ and Ω_j $(j=1,\dots,m)$ are skew-symmetric matrices, $X \in \mathrm{SO}(n)$ holds on U.*

Proof. The latter half is a direct conclusion of Proposition 2.2. We show the existence of X: Take a smooth path $\gamma\colon [0,1] \to U$ joining P_0 and P. Then by Theorem 1.9, there exists unique C^∞-map $\hat{X}\colon [0,1] \to \mathrm{M}_m(\mathbb{R})$ satisfying (2.3) with initial condition $\widetilde{X}(0) = X_0$.

We shall show that the value $\hat{X}(1)$ does not depend on choice of paths joining P_0 and P. To show this, choose another smooth path $\tilde{\gamma}$ joining P_0 and P. Since U is simply connected, there exists a homotopy between γ and $\tilde{\gamma}$, that is, there exists a continuous map $\sigma_0\colon [0,1]\times[0,1]\ni(t,w)\mapsto\sigma(t,w)\in U$ satisfying

$$\begin{aligned} \sigma_0(t,0)&=\gamma(t), & \sigma_0(t,1)&=\tilde{\gamma}(t),\\ \sigma_0(0,w)&=\mathrm{P}_0, & \sigma_0(1,w)&=\mathrm{P}. \end{aligned} \tag{2.7}$$

Then, by Whitney's approximation theorem (Theorem 10.16 in [2-3]) again, there exists a smooth map $\sigma\colon[0,1]\times[0,1]\to U$ satisfying the same boundary conditions as (2.7). We set T and W as in (2.6). For each fixed $w\in[0,1]$, there exists $X_w\colon[0,1]\to \mathrm{M}_m(\mathbb{R})$ such that

$$\frac{dX_w}{dt}(t)=X_w(t)T(t,w),\qquad X_w(0)=X_0.$$

Since $T(t,w)$ is smooth in t and w, the map

$$\check{X}(t,w)\colon[0,1]\times[0,1]\ni(t,w)\mapsto X_w(t)\in\mathrm{M}_m(\mathbb{R})$$

is a smooth map. To show that $\hat{X}(1)=\check{X}(t,0)$ does not depend on choice of paths, it is sufficient to show that b

$$\frac{\partial\hat{X}}{\partial w}=\hat{X}W \tag{2.8}$$

holds on $[0,1]\times[0,1]$. In fact, by (2.7), $W(1,w)=0$ for all $w\in[0,1]$, and then (2.8) implies that $\check{X}(1,w)$ is constant.

We prove (2.8): By definition, it holds that

$$\frac{\partial\hat{X}}{\partial t}=\hat{X}T,\qquad \hat{X}(0,w)=X_0 \tag{2.9}$$

for each $w\in[0,w]$. Hence by (2.5),

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial\hat{X}}{\partial w}&=\frac{\partial^2\hat{X}}{\partial t\partial w}=\frac{\partial^2\hat{X}}{\partial w\partial t}=\frac{\partial}{\partial w}\hat{X}T\\ &=\frac{\partial\hat{X}}{\partial w}T+\hat{X}\frac{\partial T}{\partial w}=\frac{\partial\hat{X}}{\partial w}T+\hat{X}\left(\frac{\partial W}{\partial t}+TW-WT\right)\\ &=\frac{\partial\hat{X}}{\partial w}T+\hat{X}\frac{\partial W}{\partial t}+\frac{\partial\hat{X}}{\partial t}W-\hat{X}WT\\ &=\frac{\partial}{\partial t}(\hat{X}W)+\left(\frac{\partial\hat{X}}{\partial w}-\hat{X}W\right)T.\end{aligned}$$

So, the function $Y_w(t):=\partial\hat{X}/\partial w-\hat{X}W$ satisfies the ordinary differential equation

$$\frac{dY_w}{dt}(t)=Y_w(t)T(t,w),\quad Y_w(0)=O$$

holds for each $w\in[0,1]$. Thus, by the uniqueness of the solution, $Y_w(t)=O$ holds on $[0,1]\times[0,1]$. Hence we have (2.8).

Thus, $\hat{X}(1)$ depends only the end point P of the path. Hence we can set $X(\mathrm{P}):=\hat{X}(1)$ for each $\mathrm{P}\in U$, and obtain a map $X\colon U\to\mathrm{M}_m(\mathbb{R})$. Finally we show that X is the desired solution. The initial condition $X(\mathrm{P}_0)=X_0$ is obviously satisfied. On the other hand, if we set

$$Z(\delta):=X(u^1,\dots,u^j+\delta,\dots,u^m)-X(u^1,\dots,u^m),$$

$Z(\delta)$ satisfies the equation (2.3) for the path $\gamma(\delta) := (u^1, \dots, u^j + \delta, \dots, u^m)$ with $Z(0) = X(\mathrm{P})$. Since $\Omega_\gamma = \Omega_j$,

$$\frac{\partial X}{\partial u^j} = \frac{dZ}{d\delta} = Z\Omega_j = X\Omega_j,$$

which completes the proof. □

Application: Poincaré's lemma.

Theorem 2.6 (Poincaré's lemma)**.** *If a differential* 1*-form*

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \dots, u^m)\, du^j$$

defined on a simply connected domain $U \subset \mathbb{R}^m$ is closed, that is, $d\omega = 0$ holds, then there exists a C^∞-function f on U such that $df = \omega$. Such a function f is unique up to additive constants.

Proof. The assumption is equivalent to

$$\frac{\partial \alpha_j}{\partial u_i} - \frac{\partial \alpha_i}{\partial u_j} = 0 \qquad (1 \leqq i < j \leqq m). \tag{2.10}$$

Consider a system of linear partial differential equations with unknown ξ, a 1×1-matrix valued function (i.e. a real-valued function), as

$$\frac{\partial \xi}{\partial u^j} = \xi\alpha_j \quad (j = 1, \dots, m), \qquad \xi(u_0^1, \dots, u_0^m) = 1. \tag{2.11}$$

Then it satisfies (2.4) because of (2.10). Hence by Theorem 2.5, there exists a smooth function $\xi(u,v)$ satisfying (2.11). In particular, Proposition 1.3 yields $\xi = \det \xi$ never vanishes. Here, $\xi(u_0^1, \dots, u_0^m) = 1 > 0$ means that $\xi > 0$ holds on U. Letting $f := \log \xi$, we have the function f satisfying $df = \omega$.

Next, we show the uniqueness: if two functions f and g satisfy $df = dg = \omega$, it holds that $d(f - g) = 0$. Hence by connectivity of U, $f - g$ must be constant. □

Application: Conjugation of Harmonic functions. In this paragraph, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. It is well-known that a function

$$f\colon U \ni u + \mathrm{i}\, v \longmapsto \xi(u,v) + \mathrm{i}\, \eta(u,v) \in \mathbb{C} \qquad (\mathrm{i} = \sqrt{-1}) \tag{2.12}$$

defined on a domain $U \subset \mathbb{C}$ is *holomorphic* if and only if it satisfies the following relation, called the *Cauchy-Riemann equations*:

$$\frac{\partial \xi}{\partial u} = \frac{\partial \eta}{\partial v}, \qquad \frac{\partial \xi}{\partial v} = -\frac{\partial \eta}{\partial u}. \tag{2.13}$$

Definition 2.7. A function $f\colon U \to \mathbb{R}$ defined on a domain $U \subset \mathbb{R}^2$ is said to be *harmonic* if it satisfies

$$\Delta f = f_{uu} + f_{vv} = 0.$$

The operator Δ is called the *Laplacian.*

Proposition 2.8. *If function f in* (2.12) *is holomorphic, $\xi(u,v)$ and $\eta(u,v)$ are harmonic functions.*

Proof. By (2.13), we have

$$\xi_{uu} = (\xi_u)_u = (\eta_v)_u = \eta_{vu} = \eta_{uv} = (\eta_u)_v = (-\xi_v)_v = -\xi_{vv}.$$

Hence $\Delta\xi = 0$. Similarly,

$$\eta_{uu} = (-\xi_v)_u = -\xi_{vu} = -\xi_{uv} = -(\xi_u)_v = -(\eta_v)_v = -\eta_{vv}.$$

Thus $\Delta\eta = 0$. □

Theorem 2.9. *Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u,v)$ a C^∞-function harmonic on U[4]. Then there exists a C^∞ harmonic function η on U such that $\xi(u,v) + \mathrm{i}\,\eta(u,v)$ is holomorphic on U.*

Proof. Let $\alpha := -\xi_v\,du + \xi_u\,dv$. Then by the assumption,

$$d\alpha = (\xi_{vv} + \xi_{uu})\,du \wedge dv = 0$$

holds, that is, α is a closed 1-form. Hence by simple connectivity of U and the Poincaré's lemma (Theorem 2.6), there exists a function η such that $d\eta = \eta_u\,du + \eta_v\,dv = \alpha$. Such a function η satisfies (2.13) for given ξ. Hence $\xi + \mathrm{i}\,\eta$ is holomorphic in $u + \mathrm{i}\,v$. □

Definition 2.10. The harmonic function η in Theorem 2.9 is called the *conjugate* harmonic function of ξ.

[4]The theorem holds under the assumption of C^2-differentiablity.

The fundamental theorem for Surfaces. Let $p\colon U \to \mathbb{R}^3$ be a parametrization of a *regular surface* defined on a domain $U \subset \mathbb{R}^2$. That is, $p = p(u,v)$ is a C^∞-map such that p_u and p_v are linearly independent at each point on U. Then $\nu := (p_u \times p_v)/|p_u \times p_v|$ is the *unit normal vector field* to the surface. The matrix-valued function $\mathcal{F} := (p_u, p_v, \nu)\colon U \to \mathrm{M}_3(\mathbb{R})$ is called the *Gauss frame* of p. We set

$$\begin{aligned} ds^2 &:= E\,du^2 + 2F\,du\,dv + G\,dv^2, \\ I\!I &:= L\,du^2 + 2M\,du\,dv + N\,dv^2, \end{aligned} \tag{2.14}$$

where

$$\begin{array}{lll} E = p_u \cdot p_u & F = p_u \cdot p_v & G = p_v \cdot p_v \\ L = p_{uu} \cdot \nu & M = p_{uv} \cdot \nu & N = p_{vv} \cdot \nu. \end{array}$$

We call ds^2 (resp. $I\!I$) the *first* (resp. *second*) *fundamental form*. Note that linear independence of p_u and p_v implies

$$E > 0, \qquad G > 0 \qquad \text{and} \qquad EG - F^2 > 0. \tag{2.15}$$

Set

$$\begin{aligned} \Gamma_{11}^1 &:= \frac{GE_u - 2FF_u + FE_v}{2(EG-F^2)}, & \Gamma_{11}^2 &:= \frac{2EF_u - EE_v - FE_u}{2(EG-F^2)}, \\ \Gamma_{12}^1 &= \Gamma_{21}^1 := \frac{GE_v - FG_u}{2(EG-F^2)}, & \Gamma_{12}^2 &= \Gamma_{21}^2 := \frac{EG_u - FE_v}{2(EG-F^2)}, \\ \Gamma_{22}^1 &:= \frac{2GF_v - GG_u - FG_v}{2(EG-F^2)}, & \Gamma_{22}^2 &:= \frac{EG_v - 2FF_v + FG_u}{2(EG-F^2)}. \end{aligned} \tag{2.16}$$

and

$$A = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix} := \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}. \tag{2.17}$$

The functions Γ_{ij}^k and the matrix A are called the *Christoffel symbols* and the *Weingarten matrix*. We state the following the *fundamental theorem for surfaces*, and give a proof (for a special case) in the following section.

Theorem 2.11 (The Fundamental Theorem for Surfaces)**.** *Let $p\colon U \ni (u,v) \mapsto p(u,v) \in \mathbb{R}^3$ be a parametrization of a regular surface defined on a domain $U \subset \mathbb{R}^2$. Then the Gauss frame $\mathcal{F} := \{p_u, p_v, \nu\}$ satisfies the equations*

$$\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda, \tag{2.18}$$
$$\Omega := \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & -A_1^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & -A_1^2 \\ L & M & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} \Gamma_{21}^1 & \Gamma_{22}^1 & -A_2^1 \\ \Gamma_{21}^2 & \Gamma_{22}^2 & -A_2^2 \\ M & N & 0 \end{pmatrix},$$

where Γ_{jk}^i $(i,j,k = 1,2)$, A_l^k and L, M, N are the Christoffel symbols, the entries of the Weingarten matrix and the entries of the second fundamental form, respectively.

Theorem 2.12. *Let $U \subset \mathbb{R}^2$ be a simply connected domain, E, F, G, L, M, N C^∞-functions satisfying (2.15), and Γ_{ij}^k, A_i^j the functions defined by (2.16) and (2.17), respectively. If Ω and Λ satisfies*

$$\Omega_v - \Lambda_u = \Omega\Lambda - \Lambda\Omega, \tag{2.19}$$

there exists a parameterization $p\colon U \to \mathbb{R}^3$ of regular surface whose fundamental forms are given by (2.14). Moreover, such a surface is unique up to orientation preserving isometries of $\mathbb{R}^3$.

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Exercises

2-1 Let $\xi(u,v) = \log\sqrt{u^2+v^2}$ be a function defined on $U = \mathbb{R}^2 \setminus \{(0,0)\}$

(1) Show that ξ is harmonic on U.

(2) Find the conjugate harmonic function η of ξ on
$$V = \mathbb{R}^2 \setminus \{(u,0) \mid u \leqq 0\} \subset U.$$

(3) Show that there exists no conjugate harmonic function of ξ defined on U.

2-2 Let $\theta = \theta(u,v)$ be a smooth function on a domain $U \subset \mathbb{R}^2$ such that $0 < \theta < \pi$, and
$$ds^2 := du^2 + 2\cos\theta\, du\, dv + dv^2, \quad II := 2\sin\theta\, du\, dv.$$
Show that the condition (2.19) is equivalent to
$$\theta_{uv} = \sin\theta.$$