5 Geodesics

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Orthogonal decomposition of the tangent space. Let $M \subset \mathbb{R}^{n+1}_s$ be a non-degenerate submanifold[2]. By non-degeneracy, the restriction of the inner product of $\mathbb{R}^{n+1}_s$ to each tangent space $T_\mathrm{P}M$ is non-degenerate. Then by Theorem 2.9, the orthogonal decomposition

$$(5.1) \qquad \mathbb{R}^{n+1}_s = T_\mathrm{P}\mathbb{R}^{n+1}_s = T_\mathrm{P}M \oplus N_\mathrm{P}, \bigl(N_\mathrm{P} := (T_\mathrm{P}M)^\perp\bigr)$$

holds for each $\mathrm{P} \in M$. The restriction of the inner product $\langle\ ,\ \rangle$ of $\mathbb{R}^{n+1}_s$ to N_P is non-degenerate. We call N_P the *normal space* of M at P. According to this decomposition, each vector $\boldsymbol{v} \in T_\mathrm{P}\mathbb{R}^{n+1}_s = \mathbb{R}^{n+1}_s$ can be decomposed uniquely as

$$(5.2) \qquad \boldsymbol{v} = [\boldsymbol{v}]^\mathrm{T} + [\boldsymbol{v}]^\mathrm{N} \qquad [\boldsymbol{v}]^\mathrm{T} \in T_\mathrm{P}M, \quad [\boldsymbol{v}]^\mathrm{N} \in N_\mathrm{P}.$$

We call $[\boldsymbol{v}]^\mathrm{T}$ (resp. $[\boldsymbol{v}]^\mathrm{N}$) the *tangential part* (resp. *normal part*) of $\boldsymbol{v}$.

For the case that M is a hypersurface of $\mathbb{R}^{n+1}_s$ (that is, $\dim M = n$), the normal space is spanned by the unit normal vector, that is, $N_\mathrm{P} = \mathbb{R}\nu_\mathrm{P}$, where ν_P is the unit normal vector of M at P:

$$(5.3) \qquad \mathbb{R}^{n+1}_s = T_\mathrm{P}\mathbb{R}^{n+1}_s = T_\mathrm{P}M \oplus \mathbb{R}\nu_\mathrm{P}.$$

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[2]Not necessarily of codimension one at this moment.

In this case,

$$(5.4) \qquad [\boldsymbol{v}]^\mathrm{N} = \varepsilon \langle \boldsymbol{v}, \nu_\mathrm{P}\rangle \nu_\mathrm{P}, \qquad [\boldsymbol{v}]^\mathrm{T} = \boldsymbol{v} - [\boldsymbol{v}]^\mathrm{N}$$

hold, where $\varepsilon = \langle \nu_\mathrm{P}, \nu_\mathrm{P}\rangle \in \{-1, 1\}$.

Vector fields along curves. A *curve* on a non-degenerate submanifold M of $\mathbb{R}^{n+1}_s$ is a map

$$\gamma\colon J \ni t \longmapsto \gamma(t) \in M \subset \mathbb{R}^{n+1}_s$$

where $J \subset \mathbb{R}$ is an interval. A curve γ is said to be *of class C^r* if γ is a C^r-map as a map into $\mathbb{R}^{n+1}_s$. The following fact is a direct conclusion of the definition of differentiability of maps:

Lemma 5.1. *A map $\gamma\colon J \to M \subset \mathbb{R}^{n+1}_s$ is of class C^r as a map into a differentiable manifold M if and only if it is of class C^r as a map into $\mathbb{R}^{n+1}_s$.*

From now on, by a word *smooth*, we mean of class C^∞. For example, a smooth curve γ means a curve γ of class C^∞.

Definition 5.2. Let $\gamma\colon J \to M \subset \mathbb{R}^{n+1}_s$ be a smooth curve on M. A *smooth vector field on M along γ* is a map

$$X\colon J \ni t \longmapsto X(t) \in T_{\gamma(t)}M \subset \mathbb{R}^{n+1}_s$$

which is of class C^∞ as a map from J to $\mathbb{R}^{n+1}_s$.

Example 5.3. Let $\gamma\colon J \to M \subset \mathbb{R}^{n+1}_s$ be a smooth curve. Then

$$\dot\gamma\colon J \ni t \longmapsto \dot\gamma(t) = \frac{d\gamma}{dt}(t) \in T_{\gamma(t)}M$$

is a smooth vector field along γ, called the *velocity* vector field of the curve γ.

Definition 5.4. A smooth curve $\gamma\colon J \to M$ is said to be *regular* if $\dot\gamma(t) \neq \mathbf{0}$ for all $t \in J$. A regular curve γ is called *non-degenerate* if $\langle \dot\gamma(t), \dot\gamma(t)\rangle \neq 0$, that is, $\dot\gamma(t)$ is not a light-like vector, for each $t \in J$. When $\langle \dot\gamma, \dot\gamma\rangle > 0$ (resp. $\langle \dot\gamma, \dot\gamma\rangle < 0$), it is said to be *space-like* (resp. *time-like*).

Example 5.5. Consider

$$Q_1 := \{\boldsymbol{x} \in \mathbb{R}^4_1 \,|\, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1\}$$

as defined in (4.4) for $n = 3$ and $s = 1$. Then

$$\begin{aligned}\gamma_1(t) &:= \bigl(\sinh t, 0, 0, \cosh t\bigr),\\ \gamma_2(t) &:= \bigl(0, \cos t, \sin t, 0\bigr),\\ \gamma_3(t) &:= \bigl(t, 1, 0, t\bigr)\end{aligned}$$

are regular curves on Q_1, which are time-like, space-like and degenerate, respectively.

Lemma 5.6. *Let $\gamma\colon J \to M$ be a non-degenerate regular curve on a non-degenerate submanifold $M \subset \mathbb{R}^{n+1}_s$. Then there exists a parameter change $t = t(\tau)$ such that*

$$\bigl|\langle \gamma', \gamma'\rangle\bigr| = 1, \qquad \textit{where } \gamma'(\tau) = \frac{d\gamma}{d\tau}\bigl(t(\tau)\bigr).$$

Proof. Set

$$\tau(t) := \int_{t_0}^{t} \sqrt{|\langle \dot\gamma(u), \dot\gamma(u)\rangle|}\, du.$$

Since γ is non-degenerate, $d\tau/dt = \sqrt{|\langle \dot\gamma, \dot\gamma\rangle|} > 0$. Then there exists the inverse function $t = t(\tau)$ of $\tau(t)$, and the chain rule yields the conclusion. $\square$

We call the parameter τ as in Lemma 5.6 the *arc-length* parameter of the curve. The arc-length parameter of a time-like curve in a Lorentzian manifold M is often called the *proper time*.

Covariant derivative of vector fields along curves. Let γ be a smooth regular curve on a non-degenerate submanifold $M \subset \mathbb{R}^{n+1}_s$, and X a smooth vector field of M along γ[3]. Then we obtain a map

$$\dot X\colon J \ni t \longmapsto \dot X(t) = \frac{dX}{dt} \in \mathbb{R}^{n+1}_s,$$

which is not a vector field on M, in general.

Definition 5.7. The vector field

$$\frac{\nabla}{dt} X(t) = \nabla_{\dot\gamma(t)} X(t) := \Bigl[\dot X(t)\Bigr]^{\mathrm{T}} \in T_{\gamma(t)} M$$

of M along γ is called the *covariant derivative* of X along γ, where $[*]^{\mathrm{T}}$ denotes the tangential component as in (5.2).

Definition 5.8. The covariant derivative

$$\frac{\nabla}{dt}\dot\gamma(t) = \nabla_{\dot\gamma(t)}\dot\gamma(t) := [\ddot\gamma(t)]^{\mathrm{T}} \in T_{\gamma(t)} M \tag{5.5}$$

[3]From now on, we assume all objects are of class C^∞ and omit the word *smooth*.

of $\dot\gamma$ along γ is called the *acceleration* of the curve γ.

Lemma 5.9. *For each curve γ on M, it holds that*

$$\frac{d}{dt}\langle\dot\gamma(t),\dot\gamma(t)\rangle = 2\left\langle\nabla_{\dot\gamma(t)}\dot\gamma(t),\dot\gamma(t)\right\rangle.$$

Proof. Since $\dot\gamma(t)\in T_{\gamma(t)}M$,

$$\begin{aligned}\frac{d}{dt}\langle\dot\gamma(t),\dot\gamma(t)\rangle &= 2\langle\ddot\gamma(t),\dot\gamma(t)\rangle = 2\left\langle[\ddot\gamma(t)]^{\mathrm{T}}+[\ddot\gamma(t)]^{\mathrm{N}},\dot\gamma(t)\right\rangle\\ &= 2\left\langle[\ddot\gamma(t)]^{\mathrm{T}},\dot\gamma(t)\right\rangle\\ &= 2\left\langle\nabla_{\dot\gamma(t)}\dot\gamma(t),\dot\gamma(t)\right\rangle. \qquad\square\end{aligned}$$

Geodesics and pre-geodesics.

Definition 5.10. A regular curve γ on a non-degenerate submanifold $M\subset\mathbb{R}^{n+1}_s$ is called a *pre-geodesic* if $\nabla_{\dot\gamma}\dot\gamma$ is proportional to $\dot\gamma$, and called a *geodesic* if $\nabla_{\dot\gamma}\dot\gamma=\mathbf{0}$ holds.

Lemma 5.11. *Let γ be a pre-geodesic on a non-degenerate submanifold $M\subset\mathbb{R}^{n+1}_s$. Then there exists a parameter change $t=t(\tau)$ such that $\tilde\gamma(\tau):=\gamma\bigl(t(\tau)\bigr)$ is a geodesic.*

Proof. Problem 5-1. □

Lemma 5.12. *Let γ be a geodesic of a non-degenerate submanifold $M\subset\mathbb{R}^{n+1}_s$. Then $\langle\dot\gamma(t),\dot\gamma(t)\rangle$ does not depend on t.*

Proof. A direct conclusion of Lemma 5.9. □

Local expressions. Here, we give an expression of geodesics in the local coordinate system. Let $M\subset\mathbb{R}^{n+1}_s$ be a non-degenerate submanifold and take a local coordinate neighborhood $(U;u^1,\dots,u^m)$ of M, where $m=\dim M$[4]. Then the inclusion map $\iota\colon M\to\mathbb{R}^{n+1}_s$ induces an immersion

$$f\colon U\ni(u^1,\dots,u^m)\longmapsto f(u^1,\dots,u^m)\in M\subset\mathbb{R}^{n+1}_s, \tag{5.6}$$

here we identify the coordinate neighborhood $U\subset M$ with a region of $\mathbb{R}^m$. We call such an f a (local) *parametrization* of M. Set

$$g_{ij}:=\left\langle\frac{\partial f}{\partial u^i},\frac{\partial f}{\partial u^j}\right\rangle(=g_{ji})\qquad(i,j=1,\dots,m), \tag{5.7}$$

which is a component of the induced metric $g:=\langle\ ,\ \rangle|_{T_{\mathrm{P}}M}$w with respect to the basis

$$\left\{\left(\frac{\partial}{\partial u^1}\right)_{\mathrm{P}},\dots,\left(\frac{\partial}{\partial u^m}\right)_{\mathrm{P}}\right\}$$

of $T_{\mathrm{P}}M$ for each $\mathrm{P}\in U$. Since the induced metric is non-degenerate, the $m\times m$-matrix (g_{ij}) is a regular matrix at each point P. We denote by (g^{ij}) the inverse matrix of (g_{ij}):

$$\sum_{k=1}^m g_{ik}g^{kj}=\delta_i^j=\begin{cases}1 & (i=j)\\ 0 & (i\neq j)\end{cases} \tag{5.8}$$

[4]Here we adopt the classical notation of indices, that is, we use the superscript as u^j, instead of u_j. In this context, we distinguish superscripts and subscripts.

Lemma 5.13. *Let γ is a curve in $U \subset M$ and express*

$$\gamma(t) = f\bigl(u^1(t), \dots, u^m(t)\bigr),$$

where $f\colon U \to M$ is a local parametrization of M as in (5.6). *Then*

$$\dot\gamma = \sum_{j=1}^{m} \frac{du^j}{dt}\frac{\partial f}{\partial u^j} \tag{5.9}$$

$$\nabla_{\dot\gamma}\dot\gamma = \sum_{j=1}^{m}\left(\frac{d^2u^j}{dt^2} + \sum_{k,l=1}^{m}\Gamma^j_{kl}\frac{du^k}{dt}\frac{du^l}{dt}\right)\frac{\partial f}{\partial u^j} \tag{5.10}$$

hold, where

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{m} g^{kl}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{lj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right). \tag{5.11}$$

Proof. Problem 5-2. □

The functions Γ^k_{ij} of (5.11) are called the *Christoffel symbols* with respect to the local coordinate system $(u^1, \dots, u^m)$.

Theorem 5.14. *Let $M \subset \mathbb{R}^{n+1}_s$ be a non-degenerate submanifold and fix $\mathrm{P} \in M$. Then for each $\boldsymbol{v} \in T_{\mathrm{P}}M$, there exists a unique geodesic $\gamma_{\boldsymbol{v}} : (-\varepsilon, \varepsilon) \to M$ satisfying*

$$\gamma_{\boldsymbol{v}}(0) = \mathrm{P}, \qquad \dot\gamma_{\boldsymbol{v}}(0) = \boldsymbol{v}.$$

Proof. Take a local coordinate system $(U; u^1, \dots, u^m)$ of M around P. Then a curve $\gamma(t) = f(u^1(t), \dots, u^n(t))$ (in the parametrization as in (5.6)) is geodesic if and only if

$$\frac{d^2u^j}{dt^2} + \sum_{k,l=1}^{m}\Gamma^j_{kl}\frac{du^k}{dt}\frac{du^l}{dt} = 0 \qquad (j = 1, \dots, m) \tag{5.12}$$

because of(5.10). Let $(u^1_0, \dots, u^n_0)$ be the coordinates of the point P and

$$\boldsymbol{v} = v^1\left(\frac{\partial}{\partial u^1}\right)_{\mathrm{P}} + \dots + v^n\left(\frac{\partial}{\partial u^n}\right)_{\mathrm{P}}.$$

Then the initial condition $\gamma(0) = \mathrm{P}$, $\dot\gamma(0) = \boldsymbol{v}$ corresponds to

$$u^j(0) = u^j_0, \qquad \frac{du^j}{dt}(0) = v^j. \tag{5.13}$$

Since the Christoffel symbols Γ^k_{ij} are functions in $(u^1, \dots, u^n)$, (5.12) is a normal form of an ordinary differential equation of second order with respect to the unkowns $u^j(t)$ $(j = 1, \dots, m)$. Hence by the fundamental theorem for ordinary differential equations, we have the unique solution of (5.12) under the initial condition (5.13). □

Definition 5.15. A geodesic γ on M is said to be *complete* if the domain of definition of $\gamma(t)$ is (can be extended to) $\mathbb{R}$.

Hopf Rinow's theorem. In this subsection, we consider a connected *Riemannian* manifolds, that is, the metrics are assumed to be positive definite. For a curve $\gamma\colon J \to M$ in a

Riemannian manifold M, we define the *length* of γ by

$$\mathcal{L}(\gamma) := \int_a^b \langle \dot\gamma(t), \dot\gamma(t)\rangle^{1/2}\, dt\bigl(\geqq 0\bigr). \tag{5.14}$$

We denote by $\mathcal{C}_{\mathrm{P},\mathrm{Q}}$ the set of piecewise smooth curves joining points P and Q in M, and define

$$d(\mathrm{P},\mathrm{Q}) := \inf_{\gamma\in\mathcal{C}_{\mathrm{P},\mathrm{Q}}} \mathcal{L}(\gamma). \tag{5.15}$$

Since M is assumed to be connected, the function $d\colon M\times M\to\mathbb{R}$ is well-defined. We can prove the following, which can be found in textbooks on Riemannian geometry:

Fact 5.16. *Let d be a function defined in* (5.15). *Then*

- *d is a distance function on M,*
- *the topology on M induced by the distance d coincides with the topology of M as a manifold.*

Moreover, if the length $\mathcal{L}(\gamma)$ of the curve $\gamma\colon J\to M$ is equal to $d(\gamma(a),\gamma(b))$, γ is a pre-geodesic.

We call the function d the *distance function on M induced from the Riemannian metric.*

Fact 5.17 (The Hopf-Rinow theorem)**.** *Let M be a connected Riemannian manifold. Then the following conditions are equivalent*:

- *All geodesics in M are complete.*

- *All geodesics starting at a point $\mathrm{P}\in M$ are complete.*
- *The distance function d induced from the Riemannian metric is complete (as a distance function).*
- *All divergent path on M[5] have infinite lengths.*
- *All bounded set on M with respect to d are relatively compact.*

Moreover, if these properties are satisfied, then, for each P, $\mathrm{Q}\in M$, there exists a geodesic joining P and $\mathrm{Q}\in M$, whose length is $d(\mathrm{P},\mathrm{Q})$.

Examples

Example 5.18 ((Pseudo) Euclidean spaces)**.** Consider $\mathbb{R}^n_s$ be the submanifold of itself. Taking the canonical coordinate system $(x^1,\dots,x^n)$, the coefficients g_{ij} in (5.7) are constants. Hence all the Christoffel symbols vanish, and (5.12) turns to be $d^2u^j/dt^2 = 0$ $(j=1,\dots,n)$. Hence a curve $(u^1(t),\dots,u^n(t))$ is a geodesic if and only if all $u^j(t)$'s are a linear functions in t, that is, a geodesic is expressed as

$$\gamma(t) = t\boldsymbol{v} + \boldsymbol{p},$$

that is, a straight line with constant velocity $\boldsymbol{v}$. Since this is defined on $\mathbb{R}$, $\mathbb{R}^n_s$ is complete. In particular, when $s=0$, the

[5] A curve $\gamma\colon[0,\infty)\to M$ is called a *divergent path* if for an arbitrary compact set K in M, there exists a number t_K such that $\gamma([t_k,\infty))\subset M\setminus K$.

distance $d(\mathrm{P}, \mathrm{Q})$ is the length of the line segment joining P and Q.

Example 5.19 (The spheres.)**.** Consider $S^n = S^n(1) \subset \mathbb{R}^{n+1}$ in the Euclidean $(n+1)$-space, which coincides with Q_1 in (4.4) for $s = 0$. Let $\boldsymbol{p} \in S^n$ and identify it with its position vector, and take a unit vector $\boldsymbol{v} \in T_{\mathrm{p}} S^n$. Set

$$\gamma_{\boldsymbol{v}}(t) = (\cos t)\boldsymbol{p} + (\sin t)\boldsymbol{v}.$$

Then we have

- $\langle \gamma_{\boldsymbol{v}}(t), \gamma_{\boldsymbol{v}}(t) \rangle = 1$, that is, $\gamma_{\boldsymbol{v}}(t)$ is a curve on S^n,
- $\ddot{\gamma}_{\boldsymbol{v}}(t) = -\gamma(t)$, that is, $\ddot{\gamma}_{\boldsymbol{v}}(t)$ is perpendicular to the tangent space of S^n at $\gamma_{\boldsymbol{v}}(t)$,
- $\gamma_{\boldsymbol{v}}(0) = \boldsymbol{p}$, $\dot{\gamma}_{\boldsymbol{v}}(0) = \boldsymbol{v}$, and
- the image $\gamma_{\boldsymbol{v}}(\mathbb{R})$ of $\gamma_{\boldsymbol{v}}$ is the intersection of S^n and the plane spanned by $\{\boldsymbol{p}, \boldsymbol{v}\}$ passing through the origin.

A circle obtained as the intersection of the sphere and a plane passing through the origin is called the *great circle*. Since all geodesics are defined on $\mathbb{R}$, S^n is a complete Riemannian manifold.

Example 5.20 (The hyperbolic spaces.)**.** Consider Q_{-1} as in (4.4) for $s = 1$. Since this is disconnected, we write a connected component of it by

$$H^n = \{\boldsymbol{x} = {}^t(x_0, \dots, x_n) \in \mathbb{R}^{n+1}_1 \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0\}.$$

Let $\boldsymbol{p} \in H^n$ and take a unit vector $\boldsymbol{v} \in T_{\boldsymbol{p}} H^n$. We set

$$\gamma_{\boldsymbol{v}}(t) = (\cosh t)\boldsymbol{p} + (\sinh t)\boldsymbol{v}.$$

Then

- $\langle \gamma_{\boldsymbol{v}}(t), \gamma_{\boldsymbol{v}}(t) \rangle = -1$, that is, $\gamma_{\boldsymbol{v}}(t)$ is a curve on H^n,
- $\ddot{\gamma}_{\boldsymbol{v}}(t) = \gamma_{\boldsymbol{v}}(t)$, that is, $\ddot{\gamma}_{\boldsymbol{v}}(t)$ is perpendicular to the tangent space of H^n at $\gamma_{\boldsymbol{v}}(t)$,
- $\gamma_{\boldsymbol{v}}(0) = \boldsymbol{p}$, $\dot{\gamma}_{\boldsymbol{v}}(0) = \boldsymbol{v}$,
- the image $\gamma_{\boldsymbol{v}}(\mathbb{R})$ is the intersection of H^n and the plane spanned by $\{\boldsymbol{p}, \boldsymbol{v}\}$ passing through the origin.

Exercises

5-1 Prove Lemma 5.11.

5-2 Prove Lemma 5.13.

5-3 Verify Example 5.20.