1 Bilinear Forms

A Review of Linear Algebra.

Definition 1.1.
- A square matrix P of real components is said to be an *orthogonal matrix* if ${}^tPP = P^tP = I$ holds, where ${}^tP$ denotes the transposition of P and I is the identity matrix.
- A square matrix A is said to be (real) *symmetric matrix* if ${}^tA = A$ holds.

Fact 1.2.
- *The eigenvalues of a real symmetric matrix are real numbers, and the dimension of the corresponding eigenspace coincides with the multiplicity of the eigenvalue.*
- *Real symmetric matrices can be diagonalized by orthogonal matrices. In other words, for each real symmetric matrix A, there exists an orthogonal matrix P satisfying*

$$P^{-1}AP = {}^tPAP = \operatorname{diag}(\lambda_1, \dots, \lambda_n),$$

where $\operatorname{diag}(\dots)$ *denotes the diagonal matrix with diagonal components "$\dots$". In particular, $\{\lambda_1, \dots, \lambda_n\}$ are the eigenvalues of A counted with their multiplicity.*

In this section, V denotes an n-dimensional vector space over $\mathbb{R}$ ($n < \infty$).

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Bilinear forms and quadratic forms.

Definition 1.3. A *symmetric bilinear form* on the vector space V is a map $q\colon V \times V \to \mathbb{R}$ satisfying the following:

- For each fixed $\boldsymbol{x} \in V$, both

$$q(\boldsymbol{x}, \cdot)\colon V \ni \boldsymbol{y} \mapsto q(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R} \quad \text{and}$$
$$q(\cdot, \boldsymbol{x})\colon V \ni \boldsymbol{y} \mapsto q(\boldsymbol{y}, \boldsymbol{x}) \in \mathbb{R}$$

  are linear maps.
- For any $\boldsymbol{x}$ and $\boldsymbol{y} \in V$, $q(\boldsymbol{x}, \boldsymbol{y}) = q(\boldsymbol{y}, \boldsymbol{x})$ holds.

The *quadratic form* associated to the symmetric bilinear form q is a map $\tilde{q}\colon V \ni \boldsymbol{x} \mapsto q(\boldsymbol{x}, \boldsymbol{x}) \in \mathbb{R}$.

Lemma 1.4. *A quadratic form determines the symmetric bilinear form. In other words, two symmetric bilinear forms with common quadratic form coincide with each other.*

Proof. Let q be a symmetric bilinear form and $\tilde{q}$ the quadratic form associated to it. Since

$$\begin{aligned}\tilde{q}(\boldsymbol{x} + \boldsymbol{y}) &= q(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y}) \\ &= q(\boldsymbol{x}, \boldsymbol{x}) + q(\boldsymbol{x}, \boldsymbol{y}) + q(\boldsymbol{y}, \boldsymbol{x}) + q(\boldsymbol{y}, \boldsymbol{y}) \\ &= \tilde{q}(\boldsymbol{x}) + 2q(\boldsymbol{x}, \boldsymbol{y}) + \tilde{q}(\boldsymbol{y})\end{aligned}$$

holds for each $\boldsymbol{x}$, $\boldsymbol{y} \in V$, we have

$$q(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}\bigl(\tilde{q}(\boldsymbol{x} + \boldsymbol{y}) - \tilde{q}(\boldsymbol{x}) - \tilde{q}(\boldsymbol{y})\bigr).$$

Hence the symmetric bilinear form q is determined by $\tilde{q}$. $\square$

By virtue of Lemma 1.4, a symmetric bilinear form itself is often called a *quadratic form*.

Example 1.5. For an $n \times n$ symmetric matrix $A = (a_{ij})$ with real components and column vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}} \in \mathbb{R}^n$, we set

$$q_A \colon \mathbb{R}^n \times \mathbb{R}^n \ni (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \longmapsto {}^t\hat{\boldsymbol{x}} A \hat{\boldsymbol{y}} \in \mathbb{R}, \tag{1.1}$$

where ${}^t\hat{\boldsymbol{x}}$ the column vector obtained by transposing $\hat{\boldsymbol{x}}$. Then q_A is a symmetric bilinear form on $\mathbb{R}^n$. In particular, q_I is the canonical inner product of $\mathbb{R}^n$, where I is the identity matrix.

Conversely, for each symmetric bilinear form q in $\mathbb{R}^n$, there exists a symmetric matrix A such that $q = q_A$. In fact, setting $a_{ij} := q(\boldsymbol{e}_i, \boldsymbol{e}_j)$, $A = (a_{ij})$ satisfies $q = q_A$, where $[\boldsymbol{e}_j]$ is the canonical basis of $\mathbb{R}^n$.

Matrix representation of quadratic forms. Take a basis $[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$ of the vector space V. Then

$$V \ni \boldsymbol{x} \longmapsto \hat{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \tag{1.2}$$
$$\left(\boldsymbol{x} = x_1\boldsymbol{v}_1 + \dots + x_n\boldsymbol{v}_n = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n] \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right)$$

gives an isomorphism of vector spaces. We call $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ the *component* of $\boldsymbol{x}$ with respect to the basis $[\boldsymbol{v}_j]$.

Lemma 1.6. *For a symmetric bilinear form q on V, there exists the unique $n \times n$ symmetric matrix A satisfying*

$$q(\boldsymbol{x}, \boldsymbol{y}) \Big(= q_A(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \Big) = {}^t\hat{\boldsymbol{x}} A \hat{\boldsymbol{y}},$$

where $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}} \in \mathbb{R}^n$ are the components of $\boldsymbol{x}$, $\boldsymbol{y}$ with respect to $[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$, respectively

Proof. Setting $A = (a_{ij})$ by $a_{ij} := q(\boldsymbol{v}_i, \boldsymbol{v}_j)$, the conclusion follows. In fact,

$$\begin{aligned} q(\boldsymbol{x}, \boldsymbol{y}) &= q\left(\sum_{i=1}^n x_i \boldsymbol{v}_i, \sum_{j=1}^n y_j \boldsymbol{v}_j \right) = \sum_{i,j=1}^n x_i y_j q(\boldsymbol{v}_i, \boldsymbol{v}_j) \\ &= \sum_{i,j=1}^n x_i y_j a_{ij} = {}^t\hat{\boldsymbol{x}} A \hat{\boldsymbol{y}} = q_A(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}), \end{aligned}$$

where $\hat{\boldsymbol{x}} = {}^t(x_1, \dots, x_n)$ and $\hat{\boldsymbol{y}} = {}^t(y_1, \dots, y_n)$. $\square$

We call the matrix A in Lemma 1.6 the *representative matrix* of q with respect to the basis $[\boldsymbol{v}_j]$.

Lemma 1.7. *Take two bases $[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$, $[\boldsymbol{w}_1, \dots, \boldsymbol{w}_n]$ of V and let $U = (u^{ij}) \in \mathrm{GL}(n, \mathbb{R})$ be the basis change matrix:*

$$[\boldsymbol{w}_1, \dots, \boldsymbol{w}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n] U$$
$$\text{i.e.,} \quad \boldsymbol{w}_j = \sum_{i=1}^n u_{ij} \boldsymbol{v}_i \quad (j = 1, \dots, n),$$

where $\mathrm{GL}(n,\mathbb{R})$ *denotes the* general linear group*, that is, the set of $n\times n$ regular matrix of real components. Let A (resp. $\tilde{A}$) be the representative matrix of a symmetric bilinear form q with respect to the basis $[\boldsymbol{v}_j]$ (resp. $[\boldsymbol{w}_j]$). Then it holds that*

$$\tilde{A} = {}^tUAU.$$

Proof. Writing $\boldsymbol{x}$ and $\boldsymbol{y}$ as

$$\boldsymbol{x} = [\boldsymbol{v}_1,\dots,\boldsymbol{v}_n]\hat{\boldsymbol{x}} = [\boldsymbol{w}_1,\dots,\boldsymbol{w}_n]\tilde{\boldsymbol{x}},$$
$$\boldsymbol{y} = [\boldsymbol{v}_1,\dots,\boldsymbol{v}_n]\hat{\boldsymbol{y}} = [\boldsymbol{w}_1,\dots,\boldsymbol{w}_n]\tilde{\boldsymbol{y}},$$

we have

$$\hat{\boldsymbol{x}} = U\tilde{\boldsymbol{x}}, \qquad \hat{\boldsymbol{y}} = U\tilde{\boldsymbol{y}}.$$

Hence $q(\boldsymbol{x},\boldsymbol{y}) = {}^t\hat{\boldsymbol{x}}A\hat{\boldsymbol{y}} = {}^t\tilde{\boldsymbol{x}}{}^tUAU\tilde{\boldsymbol{y}}$. □

Non-degenerate quadratic forms.

Definition 1.8. A symmetric bilinear form (a quadratic form) q on V is said to be

- *positive definite* (resp. *positive semi definite*) if $q(\boldsymbol{x},\boldsymbol{x}) > 0$ (resp. $\geqq 0$) holds for all $\boldsymbol{x}\in V\setminus\{\boldsymbol{0}\}$,
- *negative definite* if $-q$ is positive definite,
- *non-degenerate* when "$q(\boldsymbol{x},\boldsymbol{y}) = 0$ for all $\boldsymbol{y}\in V$" implies "$\boldsymbol{x} = \boldsymbol{0}$".

Example 1.9. An inner product (in the undergraduate Linear Algebra course) is nothing but a positive definite quadratic form.

Remark 1.10. A positive (resp. negative) quadratic form is non-degenerate. In fact, if $q(\boldsymbol{x},\boldsymbol{y}) = 0$ holds for all $\boldsymbol{y}$, $q(\boldsymbol{x},\boldsymbol{x}) = 0$ holds. On the other hand, $q(\boldsymbol{x},\boldsymbol{x}) > 0$ (resp. < 0) when $\boldsymbol{x}\neq\boldsymbol{0}$. Hence $\boldsymbol{x} = \boldsymbol{0}$.

Signature of non-degenerate quadratic forms.

Proposition 1.11. *A quadratic form q on V is positive definite (resp. positive semi-definite, negative definite, non-degenerate) if and only if all eigenvalues of the representative matrix of q are positive (resp. non-negative, negative, non-zero). This condition does not depend on choice of bases.*

Proof. Let $[\boldsymbol{v}_1,\dots,\boldsymbol{v}_n]$ be a basis of V and A the representative matrix of q with respect to it. Since A is a symmetric matrix, there exists an orthogonal matrix P such that

$${}^tPAP = \Lambda, \quad \Lambda := \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix},$$
$${}^tPP = I = \text{the identity matrix},$$

where $\lambda_1,\dots,\lambda_n$ are eigenvalues of A, which are real numbers. Then, by setting $[\boldsymbol{w}_1,\dots,\boldsymbol{w}_n] := [\boldsymbol{v}_1,\dots,\boldsymbol{v}_n]P$, the representative matrix of q with respect to $[\boldsymbol{w}_j]$ is the diagonal matrix Λ. Denoting the components of vectors $\boldsymbol{x}$,$\boldsymbol{y}$ with respect to $[\boldsymbol{w}_j]$ by $\hat{\boldsymbol{x}} = {}^t(x_1,\dots,x_n)$ and $\hat{\boldsymbol{y}} = {}^t(y_1,\dots,y_n)$, respectively,

$$q(\boldsymbol{x},\boldsymbol{y}) = \sum_{j=1}^{n}\lambda_j x_j y_j.$$

The conclusion is obtained by this equality. In fact, if q is positive (resp. negative) definite, $q(\boldsymbol{w}_j, \boldsymbol{w}_j) = \lambda_j$ is positive (resp. negative) for each $j = 1, \dots, n$. Hence all eigenvalues of A are positive (resp. negative). Conversely, if all eigenvalues are positive (resp. negative),

$$q(\boldsymbol{x}, \boldsymbol{x}) = \sum_{j=1}^{n} \lambda_j (x_j)^2$$

is positive (resp. negative). The conclusion for positive semi-definite case is obtained in the same way.

On the other hand, let q be a non-degenerate quadratic form and assume $\lambda_j = 0$ for some $j = 1, \dots, n$. Then

$$q(\boldsymbol{w}_j, \boldsymbol{w}_j) = \lambda_j \neq 0,$$

contradiction to non-degeneracy. Conversely, assume $\lambda_j \neq 0$ $(j = 1, \dots, n)$, and $q(\boldsymbol{x}, \boldsymbol{y}) = 0$ for all $\boldsymbol{y} \in V$. Then

$$0 = q(\boldsymbol{x}, \boldsymbol{w}_j) = \lambda_j x_j$$

holds, which implies $x_j = 0$, for $j = 1, \dots, n$. Thus, $\boldsymbol{x} = 0$.b Hence q is non-degenerate. □

Let W be a linear subspace of the vector space V. Then a symmetric bilinear form $q\colon V \times V \to \mathbb{R}$ on V induces a symmetric bilinear form $q|_W\colon W \times W \to \mathbb{R}$ on W.

Definition 1.12. For a non-degenerate quadratic form q on V, we define

$$n_+ := \max\{\dim W \,;\, q|_W \text{ is positive definite}\},$$
$$n_- := \max\{\dim W \,;\, q|_W \text{ is negative definite}\}.$$

The pair (n_+, n_-) is called the *signature* of q.

Example 1.13. A positive (resp. negative) definite quadratic form q on V has signature $(n, 0)$ (resp. $(0, n)$), where $n = \dim V$.

Theorem 1.14. *Let (n_+, n_-) be the signature of a non-degenerate quadratic form q on V. Then n_+ (resp. n_-) is the number of positive (resp. negative) eigenvalues of the representative matrix of q. In particular, $n_+ + n_- = n = \dim V$ holds.*

Proof. As seen in the proof of Proposition 1.11 we may assume that the matrix representative with respect to the basis $[\boldsymbol{w}_j]$ is a diagonal matrix Λ, without loss of generality. Since all diagonal components of Λ are non-zero, we may assume that $\lambda_1, \dots, \lambda_t$ are negative, and $\lambda_{t+1}, \dots, \lambda_n$ are positive. Then q is negative definite on the subspace generated by $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_t\}$, and hence $n_- \geqq t$. On the other hand, q is positive definite on the subspace generated by $\{\boldsymbol{w}_{t+1}, \dots, \boldsymbol{w}_n\}$, and then $n_+ \geqq n - t$:

$$n_- \geqq t, \qquad n_+ \geqq n - t. \tag{1.3}$$

Here, by definition, there exists a subspace W_+ (resp. W_-) of V such that $q|_{W_+}$ (resp. $q|_{W_-}$) is positive (resp. negative) definite and $\dim W_+ = n_+$ (resp. $\dim W_- = n_-$). Take a vector $\boldsymbol{x} \in W_+ \cap W_-$. Then $q(\boldsymbol{x}, \boldsymbol{x}) \leqq 0$ and $q(\boldsymbol{x}, \boldsymbol{x}) \geqq 0$ hold, that is, $q(\boldsymbol{x}, \boldsymbol{x}) = 0$. Noticing $q|_{W_+}$ is positive definite, $\boldsymbol{x} = \boldsymbol{0}$. Hence $W_+ \cap W_- = \{\boldsymbol{0}\}$, and then we have

$$n_+ + n_- = \dim W_+ + \dim W_- \leqq \dim V = n.$$

Therefore (1.3) yields

$$n_- \geqq t, \qquad n - n_- \geqq n - t \qquad n - n_+ \geqq t, \qquad n_+ \geqq n - t,$$

that is, $n_- = t$, $n_+ = n - t$. $\square$

Remark 1.15. By Theorem 1.14, the number of positive (resp. negative) eigenvalues of the matrix representative does not depend on choice of bases. This fact is equivalent to "the number of positive (negative) eigenvalues of a symmetric matrix A is invariant to the transformation $A \mapsto {}^tUAU$ (U is a regular matrix)".

Definition 1.16. An *inner product* on a finite dimensional vector space V is a non-degenerate quadratic (symmetric bilinear) form. A vector space with fixed inner product is called an *inner product space* or a *metric space*.

Pseudo Euclidean vector spaces. Let $s \geqq 0$, $t \geqq 0$ be integers satisfying $n := s + t \geqq 2$. Then a quadratic form

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_{s,t} := -\left(\sum_{j=1}^{s} v_j w_j\right) + \left(\sum_{k=s+1}^{s+t} v_k w_k\right) \qquad \left(\boldsymbol{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \boldsymbol{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}\right) \tag{1.4}$$

gives an inner product on $\mathbb{R}^n$ with signature (t, s). We denote by $\mathbb{R}^n_s$ such an inner product space, and call the *pseudo Euclidean vector space* of signature (s, t). The inner product (1.4) can be expressed as

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_{s,t} = {}^t\boldsymbol{v} J_{s,t} \boldsymbol{w} \qquad J_{s,t} := \begin{pmatrix} -I_s & O \\ O & I_t \end{pmatrix}. \tag{1.5}$$

In particular, the case of signature $(n, 0)$, $\mathbb{R}^n := \mathbb{R}^n_0$ is called the *Euclidean vector space*, and when the signature is $(n-1, 1)$, the space $\mathbb{R}^n_1$ is called the *Minkowski vector space*.

Orthonormal basis In this paragraph, we fix an inner product $\langle\ ,\ \rangle$ on V.

Definition 1.17. A vector $\boldsymbol{v} \in V$ is said to be *orthogonal* to $\boldsymbol{w} \in V$ if $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$ holds.

Definition 1.18. An n-tuple $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ of V is called an *orthonormal basis* of V if

$$|\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle| = \delta_{ij} \qquad (1 \leqq i, j \leqq n)$$

holds.

Lemma 1.19. *An orthonormal basis is a basis of V.*

Proof. It is sufficient to show linear independency. $\square$

Theorem 1.20. *There exists an orthonormal basis for an arbitrary inner product space. In particular, if the signature of the inner product is (t, s), one can take a basis $[\boldsymbol{e}_j]$ satisfying*

$$\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = 0 \quad (i \neq j),$$
$$\langle \boldsymbol{e}_i, \boldsymbol{e}_i \rangle = \begin{cases} -1 & (i = 1, \dots, s) \\ 1 & (i = s+1, \dots, s+t). \end{cases}$$

Proof. As seen in the proof of Proposition 1.11, there exists a basis $[\boldsymbol{w}_j]$ such that the matrix representative of $\langle\ ,\ \rangle$ is an diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Since the number of positive (resp. negative) eigenvalues is t (resp. s), we may assume

$$\lambda_j \begin{cases} < 0 & (j = 1, \dots, s) \\ > 0 & (j = s+1, \dots, n) \end{cases}$$

without loss of generality. We set

$$U := \operatorname{diag}\left(1/\sqrt{|\lambda_1|}, \dots, 1/\sqrt{|\lambda_n|}\right).$$

Then it holds that

$$ {}^t U \Lambda U = \begin{pmatrix} -I_s & O \\ O & I_t \end{pmatrix}, \tag{1.6}$$

where I_m is the $m \times m$ identity matrix, and O denotes the zero matrix of an appropriate size. Hence by Lemma 1.7, the matrix representative of $\langle\ ,\ \rangle$ with respect to the basis $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] := [\boldsymbol{w}_1, \dots, \boldsymbol{w}_n]U$ is the matrix in (1.6). Hence $[\boldsymbol{e}_j]$ satisfies the desired property. □

Exercises

1-1 For an $m \times n$ matrix C, we set $A := {}^t CC$, which is an $n \times n$-symmetric matrix. Let q_A be the quadratic form on $\mathbb{R}^n$ as in Example 1.5 induced from A.

(1) Prove that q_A is positive semi-definite.

(2) Find a condition of C for q_A to be positive definite.

1-2 Let $\mathrm{M}_2(\mathbb{R})$ be the set of 2×2 real matrices, and

$$\operatorname{Sym}(2, \mathbb{R}) := \{A \in \mathrm{M}_2(\mathbb{R})\,;\, {}^t A = A\},$$
$$\operatorname{Sym}_+(2, \mathbb{R}) := \{A \in \operatorname{Sym}(2, \mathbb{R})\,;\, q_A \text{ is positive definite}\},$$

where q_A is the quadratic form as defined in Example 1.5. Is the subset $\operatorname{Sym}_+(2, \mathbb{R})$ a smooth submanifold of $\mathrm{M}_2(\mathbb{R}) = \mathbb{R}^4$?