IEE. B402 Advanced Macroeconomics

Ramsey–Cass–Koopmans Model (1)

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Plan

- Setup of the model
  - Households and firms
  - Closing the model
- Competitive equilibrium path
  - Definition
  - Steady state
  - Transitional dynamics
- The social planner's problem

Introduction

- Recall that in the Solow-Swan model, consumption C and savings S are proportional to current income Y:

$$S = sY, \; C = (1 - s)Y,$$

  where $s \in (0, 1)$ is the saving rate which is assumed to be exogenous.

- Ramsey-Cass-Koopmans model (or simply, Ramsey model):
  - This model differs from the Solow-Swan model in the respect that it endogenizes the savings rate by explicitly modeling the consumer's infinite-horizon dynamic optimization.

Model

Households: Demographics and Utility Function

Population growth:

$$\dot{L}(t)(\equiv dL(t)/dt) = nL(t) \Leftrightarrow L(t) = L(0)e^{nt}. \tag{1}$$

$(*)$ $L(0)$ is normalized to one.

Lifetime utility function:

$$\begin{aligned} U &= \int_0^{\infty} e^{-\rho t} L(t) u(c(t)) dt \\ &= \int_0^{\infty} e^{-(\rho - n)t} u(c(t)) dt. \end{aligned} \tag{2}$$

Assumption 1

$\rho > n$

Households

Let $A(t)$ denote asset holdings by the representative household at t.

Flow budget constraint:

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - c(t)L(t). \tag{3}$$

- $r(t)$: interest rate; $w(t)$: wage rate.

($*$) We will discuss the relationship between r and R soon later.

Define a as follows:

$$a(t) \equiv \frac{A(t)}{L(t)}.$$

Then, we obtain the flow budget equation in per-captia terms:

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t). \tag{4}$$

Utility Maximization Problem

The representative household's utility maximization problem

$$
\begin{aligned}
\max_{(a(t),c(t))_{t\geq 0}} \quad & U = \int_0^\infty e^{-(\rho-n)t} u(c(t))dt \\
\text{s.t.} \quad & \dot{a}(t) = (r(t)-n)a(t) + w(t) - c(t), \\
& \lim_{t\to\infty} a(t) \exp\left(-\int_0^t (r(s)-n)ds\right) \geq 0, \\
& a(0) \text{ given.}
\end{aligned}
$$

Euler Equation and Transversality Condition

Euler equation:

$$\sigma(c(t))\frac{\dot{c}(t)}{c(t)} = r(t) - \rho. \tag{8}$$

where

$$\sigma(c) \equiv -\frac{cu''(c)}{u'(c)} > 0 \forall c.$$

TVC:

$$\lim_{t\to\infty} a(t)\exp\left(-\int_0^t (r(s)-n)ds\right) = 0. \tag{9}$$

Firms

A representative firm's profit maximization problem:

$$\max_{K(t),L(t)} F(K(t), L(t)) - R(t)K(t) - w(t)L(t),$$

Since $F(K, L) = f(k)L$, the above problem can be converted to

$$\max_{k(t),L(t)} [f(k(t)) - R(t)k(t) - w(t)]L(t)$$

Competitive factor markets then imply:

$$R(t) = f'(k(t)), \tag{10}$$

and

$$w(t) = f(k(t)) - k(t)f'(k(t)). \tag{11}$$

No-Arbitrage Condition

Each household has the opportunity to hold the following two types of assets

1. capital $k(t)$
2. individual bonds $b_p(t)$

(∗) When introduce the government's activity, public bonds are added.

The "no-arbitrage-conation" btw the two assets:

$$R(t) - \delta = r(t) \tag{12}$$

Market–Clearing Conditions

- Let $B_p(t)$ denote the aggregate amount of individual bonds $(A(t) = K(t) + B_p(t))$
- Since the "lending&borrowing" in an individual sense is cancelled out as a whole,

$$B_p(t) = 0.$$

- Then, asset market-clearing condition in *per capita terms* is given by

$$a(t) = k(t). \tag{14}$$

Competitive equilibrium path

Definition

- The zero-profit-condition of firms implies

$$R(t)k(t) + w(t) = f(k(t))$$

$\Downarrow$

- The household's budget constraint is rewritten as

$$\begin{aligned}\dot{k}(t) &= (R(t) - \delta - n)k(t) + w(t) - c(t) \\ &= f(k(t)) - (n + \delta)k(t) - c(t) \qquad (16)\end{aligned}$$

which is equivalent to the market-clearing condition for the final good.

Definition

- Since $r(t) = R(t) - \delta = f'(k(t)) - \delta$, the Euler eq. becomes

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma(c(t))} \left(f'(k(t)) - \delta - \rho\right) \tag{17}$$

- TVC is now expressed as

$$\lim_{t \to \infty} k(t) \exp\left(-\int_0^t [f'(k(s)) - (n + \delta)] ds\right) = 0 \tag{18}$$

Definition

Definition 1 (Competitive Equilibrium Path)

Given $k(0) > 0$, the pair of paths $(k(t), c(t))_{t \geq 0}$ which jointly satisfy (16)–(18) constitute a competitive equilibrium path.

- Eqs. (16) – (18) :
  $\rightarrow$ The system of differential equations wrt $k(t)$ and $c(t)$ with the terminal condition

$\rightarrow$ These equations are called the Autonomous Dynamic System

How can we derive the competitive equilibrium path?

Basically we use the following procedure:

1. Examine the existence and uniqueness of the Steady State
   - SS= the path in which $k(t)$ and $c(t)$ are constant.
2. Examine the stability of the steady state:
3. Check whether or not the Transitional Dynamics to the steady state is uniquely determined or not.

Existence and Uniqueness of Steady State

Steady State

- Let k^* and c^* respectively denote the steady-state values of $k(t)$ and $c(t)$.
- From the Euler equation (17) with $\dot{c} = 0$,

$$\frac{\dot{c}(t)}{c(t)} = 0 \Leftrightarrow \frac{1}{\sigma(c(t))}(f'(k(t)) - \delta - \rho) = 0.$$

- Then, k^* is determined as

$$f'(k^*) = \rho + \delta > n + \delta. \tag{19}$$

($*$) Eq. (19) pins down the steady-state capital-labor ratio only as a function of the production function, the discount rate and the depreciation rate

Steady State

- From (16) with $\dot{k} = 0$, we obtain

$$c^* = C(k^*) \equiv f(k^*) - (n + \delta)k^*. \tag{20}$$

- Note that function $C(k)$ satisfies

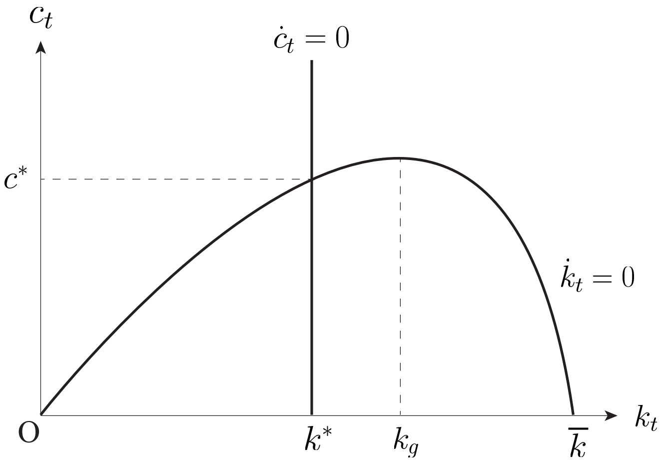
$$C'(k) = f'(k) - (n + \delta) \gtreqless 0 \Leftrightarrow f'(k) \gtreqless n + \delta.$$

- Let us define k_g such that $f'(k_g) = n + \delta$.
  k_g is golden rule of capital stock.

- Since $f'' < 0$,

$$\frac{dc^*}{dk^*} \gtreqless 0 \Leftrightarrow k \lesseqgtr k_g.$$

Steady State



Modified Golden Rule

- k^* is called the modified golden rule, which is smaller than the golden rule k_g.
  Quiz: show it.

- This implies that achieving the golden rule is not desirable from the viewpoint of utility maximizing.

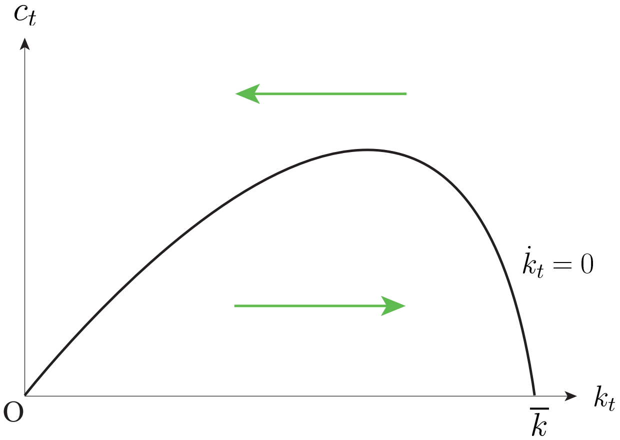
Stability of SS and transitional dynamics

Analysis (1): Graphical analysis using Phase Diagram

Transitional Dynamics

From the dynamics of $k(t)$,

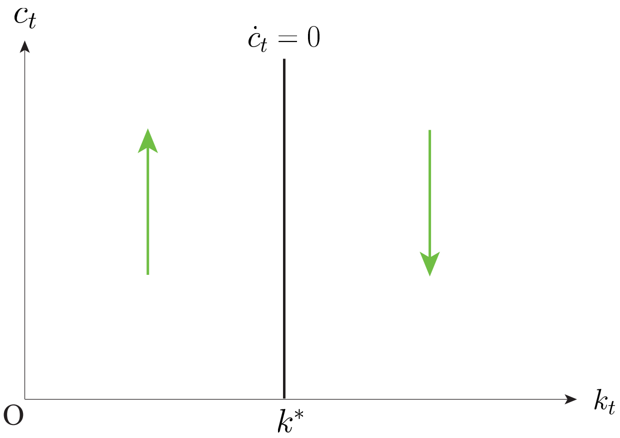
$$\dot{k}(t) \gtreqless 0 \Leftrightarrow c(t) \lesseqgtr f(k(t)) - (n+\delta)k(t) \equiv C(k(t)).$$



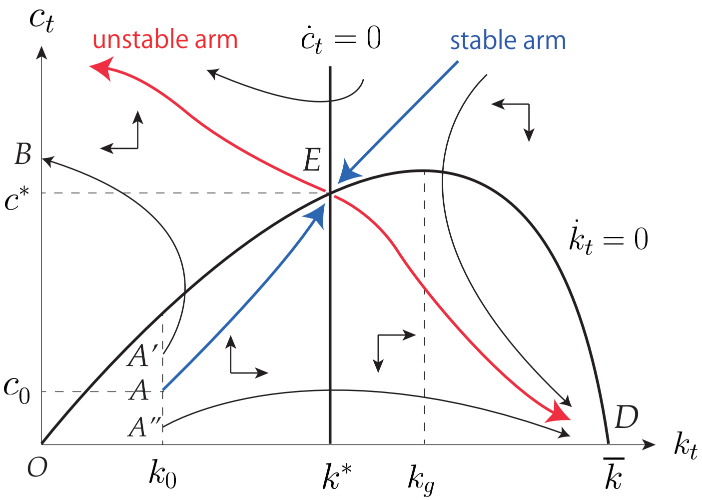
Transitional Dynamics

From the dynamics of $c(t)$ with $c(t) > 0$,

$$\dot{c}(t) \gtreqless 0 \Leftrightarrow k(t) \lesseqgtr k^*.$$



Transitional Dynamics



Stability of SS and transitional dynamics

Analysis (2): Analysis using Linear Approximation of System (16)–(17)

Local Stability of Linearized System

- Linear approximation of (16) in the neighborhood of SS:

$$\begin{aligned}\dot{k}(t) &= [f'(k^*) - (n+\delta)](k(t) - k^*) - (c(t) - c^*) \\ &= (\rho - n)(k(t) - k^*) - (c(t) - c^*)\end{aligned}$$

- Linear approximation of (17) in the neighborhood of SS:

$$\begin{aligned}\dot{c}(t) &= \frac{c^* f''(k^*)}{\sigma(c^*)}(k(t) - k^*) \\ &= -\frac{u'(c^*) f''(k^*)}{u''(c^*)}(k(t) - k^*)\end{aligned}$$

Local Stability of Linearized System

$\therefore$ The linearized (or local) dynamics:

$$\begin{pmatrix} \dot{k}(t) \\ \dot{c}(t) \end{pmatrix} = J \begin{pmatrix} k(t) - k^* \\ c(t) - c^* \end{pmatrix}$$

where J is Jacobian matrix:

$$J = \begin{pmatrix} \rho - n & -1 \\ \\ -\dfrac{f''(k^*)u'(c^*)}{u''(c^*)} & 0 \end{pmatrix}$$

Characteristic Equation

- Let ω_j $(= 1, 2)$ denote the eigenvalue of matrix J.
- ω_j is determined from the following *characteristic equation:*

$$\det(J - \omega I) = 0 \Leftrightarrow \det \begin{pmatrix} \rho - n - \omega & -1 \\ -\dfrac{f''(k^*)u'(c^*)}{u''(c^*)} & 0 - \omega \end{pmatrix} = 0,$$

$$\Leftrightarrow \omega^2 - (\rho - n)\omega + -\frac{f''(k^*)u'(c^*)}{u''(c^*)} = 0.$$

- It is shown that there are two real eigenvalues, one negative and one positive.
- Without any loss of generality, let $\omega_1 > 0$ and $\omega_2 < 0$ respectively denote the positive and the negative eigenvalues.

Local Stability of Linearized System

It is well known that the general solution is

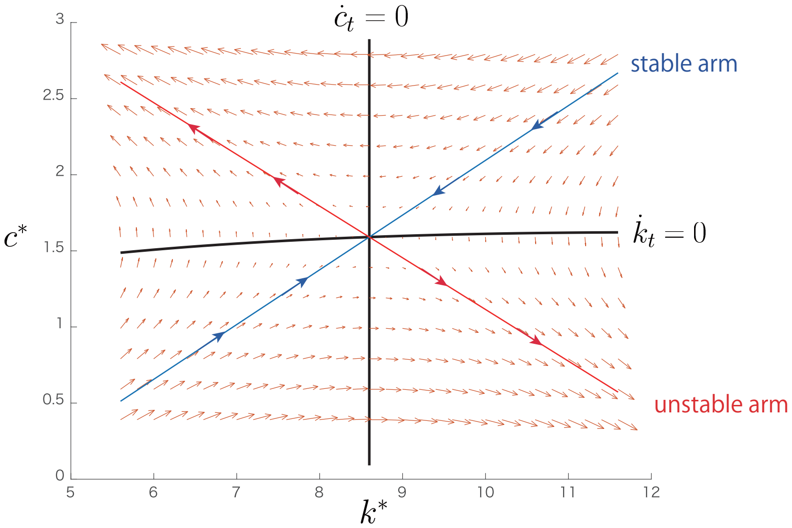
$$\begin{pmatrix} k(t) - k^* \\ c(t) - c^* \end{pmatrix} = Z_1 \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} \exp(\omega_1 t) + Z_2 \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} \exp(\omega_2 t) \quad (24)$$

- $V_j \equiv (v_{1j}, v_{2j})'$: the eigenvector corresponding to ω_j $(j \in \{1, 2\})$;
- Z_j $(j \in \{1, 2\})$: a constant value *still to be determined*

Numerical Example

- Specifications: $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, $f(k) = k^{\alpha}$
- $\alpha = 0.4$, $\delta = 0.07$, $\rho = 0.04$, $\theta = 0.1$, $n = 0.2$
  $\to \omega_1 = 0.3596$, $\omega_2 = -0.3396$
  $\to V_1 = (0.947, -0.322)'$, $V_2 = (0.941, 0.339)'$.

Numerical Example



Determination of Initial Consumption

Determination of $c(0)$:

$$\begin{aligned} k(0) - k^* &= Z_1 v_{11} + Z_2 v_{12} \text{ and} \\ c(0) - c^* &= Z_1 v_{21} + Z_2 v_{22} \end{aligned} \tag{25}$$

Thus, the initial consumption, $c(0)$, is determined such that

1. $Z_1 = 0$: otherwise the economy diverges from the steady state, and such a path violates either the Keynes-Ramsey rule or the TVC;
2. $Z_2 = (k(0) - k^*)/v_{12}$: otherwise (25) does not hold given $k(0)$.

Lemma 2

The initial consumption is determined as $c(0) = c^* + \dfrac{v_{22}}{v_{11}}(k(0) - k^*)$.

Uniqueness of Equilibrium Path

Therefore, from (24), we can analytically obtain the optimal growth path as follows:

$$
\begin{aligned}
k(t) - k^* &= (k(0) - k^*) \exp(\omega_2 t) \text{ and} \\
c(t) - c^* &= \frac{v_{22}}{v_{12}} (k(0) - k^*) \exp(\omega_2 t) \\
&\left(= \frac{v_{22}}{v_{12}} (k(t) - k^*) \right)
\end{aligned} \tag{26}
$$

Proposition 1

There exists a unique competitive equilibrium path.

The social planner's problem

Social Planner's Problem

Consider an economy in which a social planner directly determines the pair of time paths $(k(t), c(t))_{t \geq 0}$ so as to maximize

$$\max_{(k(t),c(t))_{t\geq 0}} \quad \int_0^\infty e^{-(\rho-n)t} u(c(t))dt,$$

$$\text{s.t.} \quad \dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t),$$

$$k(0) > 0 \text{ given}$$

$(*)$ This is the "original" formulation by Cass (1965) and Koopmans (1965).

The time paths $(k(t), c(t))_{t \geq 0}$ that solves the above problem is called the First-Best Allocation.

Theorem

Theorem 3

The competitive equilibrium path achieves the first-best allocation.