IEE.B402 Advanced Macroeconomics

Ramsey–Cass–Koopmans Model*

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First version: June 28, 2016

This version: June 27, 2019

Contents

*This lecture note is mainly based on chapter 8 of Acemoglu (2009). chapter 2 of Blanchard and Fischer (1989) and chapter 2 of Barro and Sala–i–Martin (2004) also provide excellent explanations of the Ramsey–Cass–Koopmans model.

1 Introduction

- Recall that in the model of Solow (1956), consumption C and savings S are proportional to current income Y:

$$S = sY,\ C = (1-s)Y,$$

where $s \in (0,1)$ is the saving rate which is assumed to be exogenous.

- **Ramsey-Cass-Koopmans model (or simply, Ramsey model):** differs from the Solow model *in the respect that the saving rate is now endogenously determined by the agents' dynamic optimizing behaviors.*

1. How much should a nation save? Ramsey (1928) discussed such a problem of "optimal saving" by explicitly modeling the decision to save.
2. Cass (1965) and Koopmans (1965) used Ramsey (1928) to extend the neoclassical growth model of Solow (1956), Swan (1956) and Phelps (1961).

Ramsey–Cass–Koopmans model (hereafter RCK model) now gives us a benchmark for many areas of modern macroeconomic analysis.

2 Setup of the Baseline RCK Model

2.1 Households

Demographics and Utility Function

The economy consists of a set of identical households. $L(t) > 0$ denotes the size of population at date $t \geq 0$. Population $L(t)$ grows at a positive constant rate of $n > 0$:

$$\dot{L}(t)(\equiv dL(t)/dt) = nL(t), \tag{1}$$

Hereafter, a dot over a variable is a time derivative of this variable. It is assumed that $L(0) = 1$. Then $L(t) = \exp(nt)$.

The lifetime utility of the representative household is given by

$$U = \int_0^\infty L(t) \times [e^{-\rho t} u(c(t))]dt, \tag{2}$$

where $c(t) \geq 0$ is per capita consumption ($c(t) = C(t)/L(t)$), $C(t) \geq 0$ is the aggregate consumption, and $\rho > 0$ is the subjective discount rate.

$u(c)$ is called the instantaneous utility function. The function u satisfies

Assumption 1. *$u(c)$ satisfies $u'(c) > 0$, $u''(c) < 0$, and $\lim_{c\to 0} u'(c) = \infty$.*

Using (1), the lifetime utility is rewritten as[1]

$$U = \int_0^\infty e^{-(\rho-n)t} u(c(t))dt. \tag{2'}$$

To ensure that there is discounting of future utility streams, we assume

Assumption 2. *$\rho > n$.*

Budget Constraint

Each household is endowed with one unit of labor, and inelastically supplies it. Therefore the population $L(t)$ corresponds to amount of labor. The aggregate budget constraint is written as

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - C(t). \tag{3}$$

where

- $A(t)$: amount of total assets held by households
- $r(t)$ and $w(t)$: the interest– and wage rate.

Defining "per-capita assets" as $a(t) \equiv A(t)/L(t)$, (3) is rewritten as

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t). \tag{4}$$

[1]Barro and Sala–i–Martin (2004) and Acemoglu (2009) assume (2) or (2') as the objective function of the representative household at date 0. Blanchard and Fischer (1989), on the other hand, assume that the sum of the representative member's instantaneous utility as her objective function:

$$U = \int_0^\infty \frac{L(t) \times [e^{-\rho t} u(c(t))]}{L(t)} dt = \int_0^\infty e^{-\rho t} u(c(t))dt.$$

Although such a difference slightly changes the household's first-order-conditions of utility maximization, key features of competitive equilibrium obtained below remain intact whichever we assume.

Maximization of Lifetime Utility

The lifetime utility maximization problem of the representative household is given by

$$\begin{aligned}
\max_{(c(t),a(t))_{t\geq 0}} \quad & U = \int_0^\infty e^{-(\rho-n)t} u(c(t))dt \\
\text{s.t.} \quad & \dot{a}(t) = (r(t)-n)a(t) + w(t) - c(t), \\
& \lim_{t\to\infty} a(t) \exp\left(-\int_0^t \Big(r(s)-n\Big)ds\right) \geq 0, \\
& a(0) \text{ given.}
\end{aligned}$$

The current-value Hamiltonian is formulated as

$$H(a(t), c(t), \lambda(t), t) = u(c(t)) + \lambda(t)\Big[(r(t)-n)a(t) + w(t) - c(t)\Big].$$

Then, the necessary and sufficient conditions of utility maximization are given by

$$\partial H/\partial c(t) = 0 \;\Leftrightarrow u'(c(t)) = \lambda(t), \tag{5}$$

$$\partial H/\partial a(t) + \dot{\lambda}(t) - (\rho - n)\lambda(t) = 0 \Leftrightarrow \dot{\lambda}(t)/\lambda(t) = \rho - r(t), \tag{6}$$

$$\lim_{t\to\infty} \lambda(t)a(t)e^{-(\rho-n)t} = 0. \tag{7}$$

From (5) and (6), we obtain the following Euler equation:

$$-\frac{c(t)u''(c(t))}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} = r(t) - \rho. \tag{8}$$

On the other hand, (7) is the transversality condition, hereafter abbreviated as TVC. From (6) we have $\lambda(t) = \lambda(0)\exp(\int_0^t(\rho - r(s))ds)$. Using this, the TVC (7) is rewritten as

$$\lim_{t\to\infty} a(t)\exp\left(-\int_0^t (r(s)-n)ds\right) = 0. \tag{9}$$

2.2 Firms

Production Function

Output, $Y(t)$, is produced using physical capital, $K(t)$, and labor, $L(t)$:

$$Y(t) = F(K(t), L(t)).$$

As in the Solow model, we are based on the following assumptions:

Assumption 3. *The production function F is twice differentiable in K and L, and satisfies the following neoclassical properties.*

1. *Positive marginal products:*

$$F_K(K, L) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \;\; F_L(K, L) \equiv \frac{\partial F(\cdot)}{\partial L} > 0.$$

2. *Diminishing marginal products:*

$$F_{KK}(K, L) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \;\; F_{LL}(K, L) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0.$$

Assumption 4. *The production function F is homogenous of degree one (or linearly homogenous):*

$$F(\lambda K, \lambda L) = \lambda F(K, L) \;\forall \lambda \geq 0.$$

Assumption 5. *The production function F satisfies the Inada conditions:*

$$\lim_{K \to 0} F_K(\cdot) = \infty, \;\; \lim_{K \to \infty} F_K(\cdot) = 0 \;\forall L > 0,$$
$$\lim_{L \to 0} F_L(\cdot) = \infty, \;\; \lim_{L \to \infty} F_L(\cdot) = 0 \;\forall K > 0.$$

Example: $F(K, L) = AK^{\alpha}L^{1-\alpha}$, where $A > 0$ and $0 < \alpha < 1$. In this case $f(k) \equiv Ak^{\alpha}$. It is easily verified that the Cobb–Douglas production function satisfies Assumptions 3–5.

Profit Maximization

A representative firm's profit maximization problem is

$$\max_{K,L} \quad F(K(t), L(t)) - R(t)K(t) - w(t)L(t),$$

where $R(t)$ is the rental rate of capital. The relationship between this and the interest rate is

$$R(t) - \delta = r(t). \tag{10}$$

Here we introduce a new variable indicating capital-labor ratio, $k \equiv K/L$. From the linear homogeneity of the function F, we can define f as the per capita production function:

$$f(k) \equiv F(k, 1).$$

Lemma 1. *Under Assumptions 3–5, f satisfies (i)$f'(k) > 0$, (ii)$f''(k) < 0$, (iv)$\lim_{k \to 0} f'(k) = \infty$, and (v)$\lim_{k \to \infty} f'(k) = 0$.*

Proof. Quiz. □

Using these properties, the profit maximization problem is rewritten as

$$\max_{k,L} \quad \left(f(k(t)) - R(t)k(t) - w(t)\right) L(t).$$

Under perfect competition, the firms takes $R(t)$ and $w(t)$ as given. The first-order conditions are

$$k(t): \quad R(t) = f'(k(t)), \tag{11}$$

$$L(t): \quad w(t) = f(k(t)) - k(t)f'(k(t)). \tag{12}$$

From (11) and (12), we can verify the zero profit condition of the firms.

$$R(t)k(t) + w(t) = f(k(t)). \tag{13}$$

2.3 Closing the Model: Market-Clearing Conditions

We consider an asset market equilibrium. The household assets, $A(t)$ can consist of

1. Capital stock, $K(t)$, which the households rent to firms,

2. Individual bonds (IOUs) due to lending and borrowing within households.

However, the individual bonds are in zero net supply here. Therefore, the asset market equilibrium is given by

$$A(t) = K(t) \Leftrightarrow a(t) = k(t). \tag{14}$$

On the other hand, the goods market equilibrium is given by

$$\underbrace{F(K(t), L(t))}_{\text{Gross Domestic Product}} = \underbrace{C(t)}_{\text{Consumption}} + \underbrace{\dot{K}(t) + \delta K(t)}_{\text{Aggregate Investment}}. \tag{15}$$

In per capita terms, we have

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t). \tag{16}$$

Caution (Walras' Law): Substituting (14) into the households' aggregate budget constraint (3) leads

$$\dot{K}(t) = r(t)K(t) + w(t)L(t) - C(t).$$

Substituting (10) into the result, we have

$$\dot{K}(t) = (R(t) - \delta)K(t) + w(t)L(t) - C(t).$$

Using (13), we rewrite the above equation as

$$\dot{K}(t) = F(K(t), L(t)) - \delta K(t) - C(t),$$

which is equivalent to (15). Thus, the goods market equilibrium (15) automatically implies from the other conditions of market equilibria and the economic agents' budget constraint.

3 Competitive Equilibrium Path

3.1 Definition

As already known, the household's budget constraint (4) is combined with (10), (13) and (14) to give (16). This equation describes the dynamics of physical capital $k(t)$, which depends on $k(t)$ and $c(t)$. On the other hand, from (10) and (11),

$$r(t) = f'(k(t)) - \delta.$$

Substituting this equation into the household's Euler equation (8), we can obtain

$$-\frac{c(t)u''(c(t))}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho. \tag{17}$$

This equation describes the dynamics of consumption $c(t)$, which depends on $k(t)$ and $c(t)$.

Using (10), (11) and (14), the TVC (9) is rewritten as

$$\lim_{t\to\infty} k(t)\exp\left(-\int_0^t (f'(k(s)) - n - \delta)ds\right) = 0. \tag{18}$$

Hereafter we refer to (18) as the TVC.

The initial condition is given by $k(0)(= a_0)$. Now we are in position to define the *Competitive Equilibrium Path.*

Definition 1 (Competitive Equilibrium Path). *Given* $k(0) > 0$*, the pair of paths* $(k(t), c(t))_{t \geq 0}$ *which jointly satisfy (16)–(18) constitute a competitive equilibrium path.*

(∗) (16) and (17) are the system of differential equations with respect to $k(t)$ and $c(t)$. These equations are called the *Autonomous Dynamic System* of the model.[2]

Both of (16) and (17) are nonlinear differential equations. How can we derive the competitive equilibrium path? Basically we use the following procedure:

1. Examine the existence and uniqueness of the *Steady State*, where a steady state is defined as an equilibrium path in which $k(t)$ and $c(t)$ are constant.

2. Examine the stability of the steady state:

   Briefly speaking, the stability refers to the ability of the steady state to converge to it when the initial value of the state variable $k(0)$ is different from the steady state.

3. Check whether or not the *Transitional Dynamics* to the steady state is uniquely determined or not.

3.2 Existence and Uniqueness of Steady State

A steady state is defined as an equilibrium path in which $k(t)$ and $c(t)$ are constant. We denote the steady state values of these variables, k^* and c^*, respectively. From (17) with $\dot{c}(t) = 0$, we have

$$f'(k^*) = \rho + \delta. \tag{19}$$

Note that from Lemma 1, there uniquely exists $k^* > 0$ that solves the above equation. Then, from (16) with $\dot{k}(t) = 0$, c^* is determined as

$$c^* = f(k^*) - (n + \delta)k^*. \tag{20}$$

Proposition 1. *There exists a unique steady state* (k^*, c^*) *that satisfies the the TVC (18) and* $k^* > 0$ *and* $c^* > 0$.

[2]Consider the system of differential equations of variables $x(t)$ and $y(t)$. Then the system is called autonomous when it depends on time only through the variables $x(t)$ and $y(t)$.

Proof. Since $k^* > 0$ has been already shown, it is sufficient to show that $c^* > 0$ and that k^* satisfies the TVC. Recall that $f(k)$ is strictly concave and $f(0) = 0$ are assumed in Assumption 5. Then, it follows that

$$f(k^*) > f(0) + f'(k^*)(k^* - 0)\ \forall k > 0\ \ \rightarrow f(k^*) > f'(k^*)k^* \forall k^* > 0$$

Substituting this result into (20) and using (19), we obtain

$$\begin{aligned} c^* > f'(k^*)k^* - (n+\delta)k^* &= [f'(k^*) - (n+\delta)]k^* \\ &= [\rho + \delta - (n+\delta)]k^* \\ &= (\rho - n)k^*. \end{aligned}$$

Since $\rho - n > 0$ holds from Assumption 2, we can show that $c^* > 0$.

Next we will show that k^* satisfies the TVC. Substituting k^* into the left-hand-side (LHS) of (18) and using (19),

$$\begin{aligned} \text{LHS of (18)} &= \lim_{t\to\infty} k^* \exp\left[-(f'(k^*) - n - \delta)t\right] \\ &= \lim_{t\to\infty} k^* \exp\left[-(\rho + \delta - n - \delta)t\right] \\ &= \lim_{t\to\infty} k^* \exp\left[-(\rho - n)t\right] \end{aligned}$$

Since $\rho - n > 0$ holds from Assumption 2 and k^* is finite, we can show the LHS converges to zero. □

Figure 1 graphically show how k^* and c^* are determined. In Figure 1, k_g is defined as

$$k_g = \arg\max_k f(k) - (n+\delta)k.$$

That is, k_g is called the *Golden Rule of Capital*.

Lemma 2. *$k^* < k_g$ under Assumption 2.*

Proof. Quiz. □

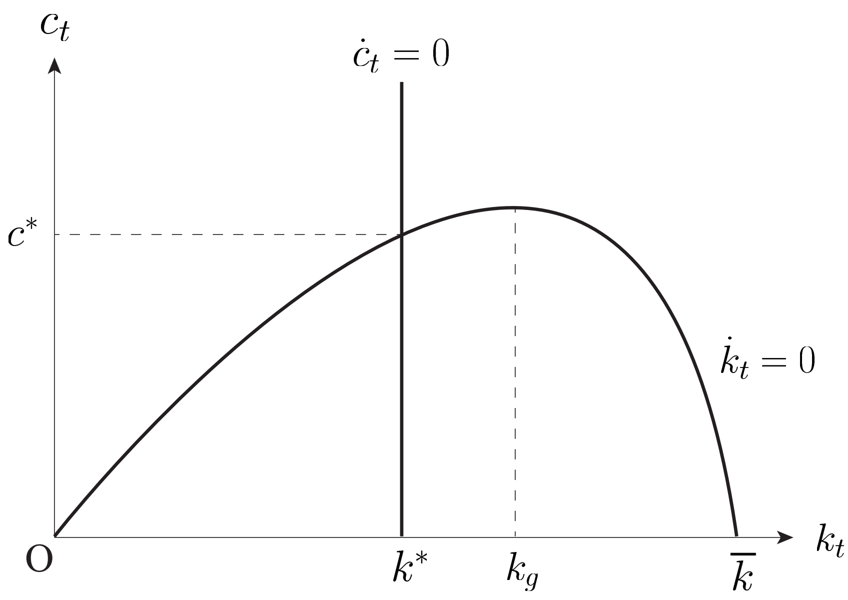

Figure 1: Steady State

3.3 Stability of Steady State and Transitional Dynamics

Since $k(0)$ is historically given, the equilibrium paths of $k(t)$ and $c(t)$ are determined, once $c(0)$ is determined. Then, how can we solve the equations for $c(0)$?

1. Graphical Solution: we depict the phase diagram.

2. Analytical Solution: we linearize the system of equations (16) and (17) in the neighborhood of the steady state, and then solve these approximated equations.

Phase Diagram

From (16) and (17),

$$\dot{k}(t) \gtreqless 0 \Leftrightarrow c(t) \lesseqgtr f(k(t)) - (n+\delta)k(t), \tag{21}$$

$$\dot{c}(t) \gtreqless 0 \Leftrightarrow -\frac{u'(c(t))}{u''(c(t))}(f'(k(t)) - \delta - \rho) \gtreqless 0. \tag{22}$$

Assumption 6. $-\dfrac{u'(c)}{u''(c)} = 0$ *if* $c = 0$.

Therefore, (22) is simplified to

$$\begin{aligned} \dot{c}(t) \gtreqless 0 \quad &\Leftrightarrow f'(k(t)) \gtreqless \rho + \delta \\ &\Leftrightarrow k(t) \lesseqgtr k^* \text{ when } c(t) > 0, \end{aligned} \tag{23}$$

and $\dot{c}(t) = 0$ for all $k(t)$ when $c(t) = 0$.

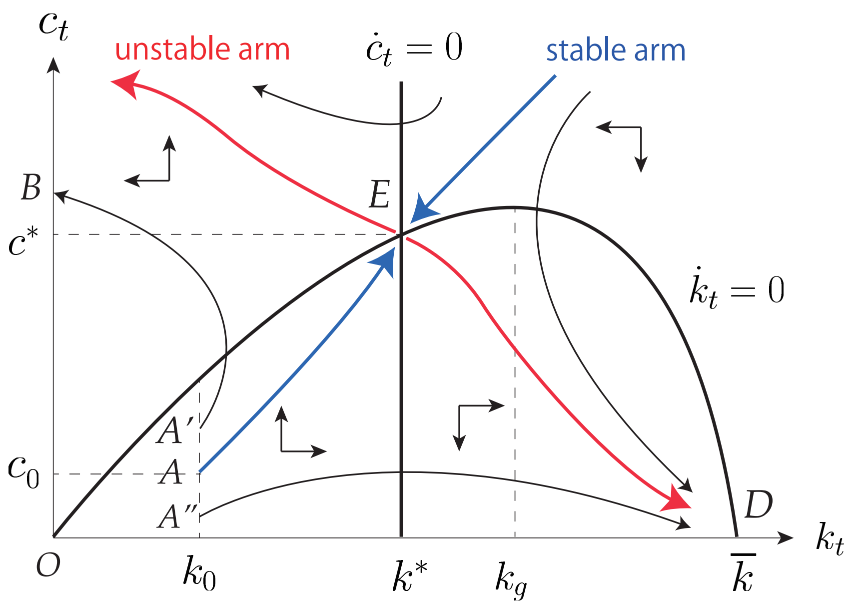

Figure 2: Phase Diagram

Figure 2 depicts the dynamics of $k(t)$ and $c(t)$ given by (21) and (23). Note that all points in the positive orthant are feasible. So there is a continuum of feasible paths. However, the paths are classified mainly into the following three types.

1. The path that converges to the steady state (k^*, c^*). Given $k(0)$, a representative shape of this type is depicted by the curve AE.

2. The paths that eventually hit the vertical axis, and move to the origin. Given $k(0)$, a representative shape of this type is depicted by the curve $A'B$. Once the path hits the point B, then it jumps to the point O.

3. The paths that converge to the steady state $(\overline{k}, 0)$, where $\overline{k}$ is the steady state capital stock with zero consumption: $f(\overline{k}) = (n + \delta)\overline{k}$. Given $k(0)$, a representative shape of this type is depicted by the curve $A''D$.

Proposition 2. *Only the path converging to the steady state (k^*, c^*) is optimal.*

Proof. We show that on all other paths, either the Euler equation (17) eventually falls or the TVC (18) is not satisfied.

At first, consider the paths which eventually hit the vertical axis. Note that on this type of paths, $\dot{c}(t) > 0$ and the sign of $\dot{k}(t)$ becomes negative sooner or later. Differentiating (16) yields

$$\left|\frac{d\dot{k}(t)}{dt}\right| = -\Big[f'(k(t)) - (n+\delta)\Big]\dot{k}(t) + \dot{c}(t) > 0.$$

This implies that these paths hit the vertical axis in *finite time*. When a path reaches the vertical axis, $k(t)$ becomes zero. Then the economy has to jump to the origin. However, such a jump violates the Keynes-Ramsey rule, which imposes $c(t)$ to *continuously* move over time. Therefore, all the paths hitting the the vertical axis cannot be optimal.

Second, consider the paths converging to the steady state $(\overline{k}, 0)$. Note that in this steady state it follows that

$$f'(\overline{k}) < n + \delta = f'(k_g).$$

The above inequality shows

$$\lim_{t\to\infty} \overline{k}\exp[(n+\delta - f'(\overline{k}))t] = \infty.$$

Thus, the steady state $(\overline{k}, 0)$ does not satisfy the TVC. □

Linear Approximation of Equilibrium System

Linearization of the dynamic system yields further insights into the dynamic behavior. A first-order Taylor expansion of (16) and (17) around the steady state (k^*, c^*) gives

$$\dot{k}(t) \simeq (\rho - n)(k(t) - k^*) - (c(t) - c^*),$$

and

$$\dot{c}(t) \simeq -\frac{f''(k^*)u'(c^*)}{u''(c^*)}(k(t) - k^*).$$

Then, the local dynamics is given by

$$\begin{pmatrix} \dot{k}(t) \\ \dot{c}(t) \end{pmatrix} = \begin{pmatrix} \rho - n & -1 \\ -f''u'/u'' & 0 \end{pmatrix} \begin{pmatrix} k(t) - k^* \\ c(t) - c^* \end{pmatrix}.$$

Let J denote the Jacobian matrix of the dynamics:

$$J \equiv \begin{pmatrix} \rho - n & -1 \\ -f''u'/u'' & 0 \end{pmatrix}.$$

So the eigenvalues of J are given by the values of ω that solve the following quadratic equation:

$$\det(J - \omega I) = 0 \Leftrightarrow \omega^2 - (\rho - n)\omega - f''u'/u'' = 0$$

The above equation is called the *Characteristic Equation.* Generally, in the two variable–two equation dynamic system, steady state is called

1. **Saddle point**, if the system has one positive- and one negative eigenvalues,
2. **Source**, if the system has two positive eigenvalues,
3. **Sink**, if the system has two negative eigenvalues.

Since $-f''u'/u'' < 0$ represents the product of the two roots and the sign of this is negative, there are two real eigenvalues, one negative and one positive.

Lemma 3. *The steady state (k^*, c^*) is a saddle point.*

Without any loss of generality, let $\omega_1 > 0$ and $\omega_2 < 0$ respectively denote the positive and the negative eigenvalues. Therefore, the general solution is given by

$$\begin{pmatrix} k(t) - k^* \\ c(t) - c^* \end{pmatrix} = Z_1 \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} \exp(\omega_1 t) + Z_2 \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} \exp(\omega_2 t), \tag{24}$$

where

- $v_j \equiv (v_{1j}, v_{2j})^T$: the eigenvector corresponding to the eigenvalue ω_j,
- Z_j: a constant value still to be determined.

Now we will show that when the steady state is a saddle point, the initial consumption is uniquely determined. Accordingly, the competitive equilibrium path is uniquely determined. Since $\omega_1 > 0$, the economy never converges to the steady state (k^*, c^*) if Z_1 would not be zero. Since (24) holds at $t = 0$, we have

$$k(0) - k^* = Z_1 v_{11} + Z_2 v_{12}, \quad c(0) - c^* = Z_1 v_{21} + Z_2 v_{22}. \tag{25}$$

Thus, the initial consumption, $c(0)$, is determined such that

1. $Z_1 = 0$: otherwise the economy diverges from the steady state, and such a path violates either the Keynes-Ramsey rule or the TVC;

2. $Z_2 = (k(0) - k^*)/v_{12}$: otherwise (25) does not hold given $k(0)$.

Lemma 4. *The initial consumption is determined as* $c(0) = c^* + \dfrac{v_{22}}{v_{11}}(k(0) - k^*)$.

Therefore, from (24), we can analytically obtain the optimal growth path as follows:

$$k(t) - k^* = (k(0) - k^*)\exp(\omega_2 t), \quad c(t) - c^* = (v_{22}/v_{12})(k(0) - k^*)\exp(\omega_2 t). \tag{26}$$

(26) is the competitive equilibrium path near around the steady state.

Proposition 3. *There exists a unique competitive equilibrium path.*

4 The First Theorem of Welfare Economics

Before going to further characterization of competitive equilibrium, let us turn to the *Social Planner's Optimal Growth Problem.* Briefly, the social planner is the agent who maximizes the representative household's life-time utility (2), and *directly* faces the dynamics of $k(t)$, (16), as the constraint of the problem. Any allocation chosen by a social planner is called the *Socially Optimal Allocation* or *Command Optimum.* By definition, this is the first-best.

The social planner's problem is given by

$$\begin{aligned} \max_{(k(t),c(t))_{t\geq 0}} \quad & U = \int_0^\infty e^{-(\rho-n)t} u(c(t))dt \\ \text{s.t.} \quad & \dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t), \\ & k(0) \geq 0 \text{ given.} \end{aligned}$$

The current–value Hamiltonian associated with the above problem is given by

$$H^{sp}(k(t), c(t), \eta(t)) = u(c(t)) + \eta(t)\left(f(k(t)) - (n+\delta)k(t) - c(t)\right).$$

Then, the optimal path is characterized by (16) and

$$\frac{\partial H^{sp}}{\partial c} = 0 \;\Leftrightarrow\; u'(c(t)) = \eta(t), \tag{27}$$

$$\frac{\partial H^{sp}}{\partial k} + \dot{\eta}(t) - (\rho - n)\eta(t) = 0 \;\Leftrightarrow\; \dot{\eta}(t) = (\rho + \delta - f'(k(t)))\eta(t), \tag{28}$$

$$\lim_{t\to\infty} \eta(t)k(t)e^{-(\rho-n)t} = 0. \tag{29}$$

From (27) and (28), we can obtain the same equation as the dynamics of $c(t)$, (17). On the other hand, from (28) we have $\eta(t) = \eta(0)\exp\left[\int_0^t (\rho + \delta - f'(k_s))ds\right]$. Substituting this into (29) gives the same condition as (18).

Proposition 4. *The competitive equilibrium path achieves the first-best allocation.*

5 Extension 1: Introducing The Public Policies

We now introduce the government into the model. Topics to be covered are (i) the effect of government spending, (2) the effect of debt financing, and (3) the effect of taxations on the equilibrium path.

5.1 Effects of Government Spending under Balanced Budget

Suppose that the government consumes $G(t)$ units of the final good. In per capita terms, $g(t) = G(t)/L(t)$. The government levies lump-sum taxes $T(t)$ to finance the expenditure. Therefore the government's budget constraint is

$$T(t)/L(t) = g(t). \tag{30}$$

We assume the path of $g(t)$ is exogenously given. Then the above equation determines the path of $T(t)$.

Equilibirum

The household's flow budget constraint now becomes

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t) - T(t)/L(t), \tag{31}$$

The household takes the path of $T(t)$ as given. Therefore the Euler equation does not change and it is found that the dynamics of $c(t)$ is essentially same as that in the economy without the government:

$$-\frac{c(t)u''(c(t))}{u'(c(t))}\dot{c}(t)/c(t) = f'(k(t)) - \delta - \rho.$$

On the other hand, the dynamics of $k(t)$ now becomes

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t) - g(t).$$

Figure 3 is the phase diagram when $g(t)$ is exogenously constant over time. Hereafter we assume that $g(t)$ is exogenously constant over time: $g(t) = g$. What happens if g increases? In steady state government spending completely crowds out private consumption, *but has no effect on the capital stock.*

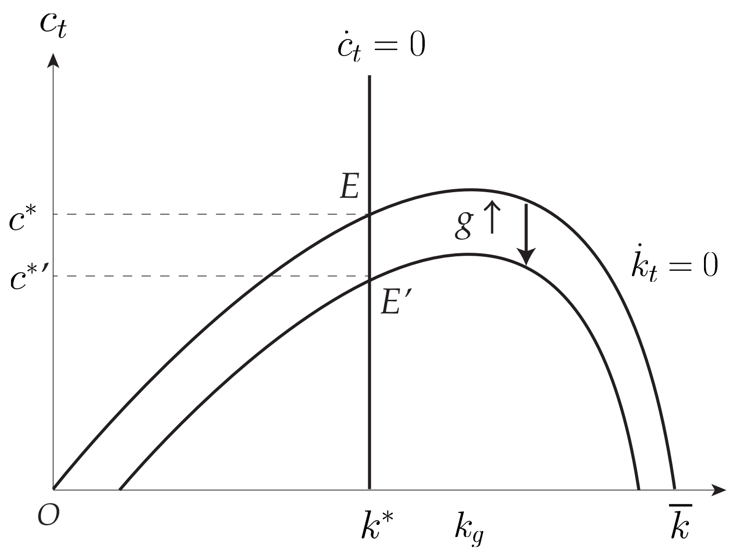

Figure 3: Effects of government spending

5.2 Effects of Debt Financing and the Ricardian Neutrality

Now relax the balanced-budget assumption (30). The government is now allowed to borrow, instead of financing itself only through lump-sum taxes. Let $B(t) \geq 0$ denote stock of government debt at date t. The government's budget constraint is now given by

$$\underbrace{T(t) + \dot{B}(t)}_{\text{Revenue}} = \underbrace{r(t)B(t) + G(t)}_{\text{Expenditure}},$$

or equivalently,

$$\dot{B}(t) = \underbrace{r(t)B(t) + \underbrace{G(t) - T(t)}_{\text{Primary deficit}}}_{\text{Budget deficit}}.$$

Integrating the above equation from zero to infinity,

$$B(0) = \int_0^\infty (T(t) - G(t)) \exp\left(-\int_0^t r_s ds\right) dt + \lim_{t\to\infty} B(t)\left(-\int_0^t r_s ds\right).$$

The no-Poinzi Game condition which prohibits the government to default is

$$\lim_{t\to\infty} B(t)\left(-\int_0^t r_s ds\right) = 0,$$

which leads the following intertemporal budget constraint of the government:

$$B(0) = \int_0^\infty (T(t) - G(t)) \exp\left(-\int_0^t r_s ds\right) dt. \tag{32}$$

The asset market equilibrium is now given by

$$A(t) = K(t) + B(t).$$

Then, the households' aggregate intertemporal budget constraint is

$$\int_0^\infty C(t) \exp\left(-\int_0^t r_s ds\right) dt = k(0) + B(0) + \int_0^\infty (w(t)L(t) - T(t)) \exp\left(-\int_0^t r_s ds\right) dt. \tag{33}$$

Then, substituting (32) into (33) yields

$$\int_0^\infty C(t) \exp\left(-\int_0^t r_s ds\right) dt = k(0) + \int_0^\infty (w(t)L(t) - G(t)) \exp\left(-\int_0^t r_s ds\right) dt. \tag{34}$$

Notice that

1. Neither taxes $T(t)$ nor the debt $B(t)$ appears in the budget constraint,
2. Only government spending $G(t)$ matters.

This result is summarized as follows:

Proposition 5. *For a given path of $G(t)$, financing it through distortionless taxation and budget deficit are indifferent.*

In other words, the method of finance, whether distortionless taxation or budget deficit has no effect on equilibrium allocation. This property is called the *Ricardian Neutrality* or *Ricardian Equivalence.*

5.3 Effects of Income Taxations

We now consider the following household' budget constraint

$$\dot{a}(t) = [(1-\tau^a)r(t) - n]a(t) + w(t) - c(t) - T(t)/L(t), \tag{35}$$

where $\tau^a \in [0, 1)$ is the capital income tax rate. It is assumed that the tax rate is constant over time.

The current–value Hamiltonian is now given by

$$H(a(t), c(t), \lambda(t), t) = u(c(t)) + \lambda(t)\big\{[(1-\tau^a)r(t) - n]a(t) + w(t) - c(t) - T(t)/L(t)\big\}. \tag{36}$$

The conditions for utility maximization are

$$\partial H/\partial c(t) = 0 \Leftrightarrow u'(c(t)) = \lambda(t), \tag{37}$$

$$\partial H/\partial a(t) = \dot{\lambda}(t) - (\rho - n)\lambda(t) \Leftrightarrow \dot{\lambda}(t)/\lambda(t) = \rho - (1-\tau^a)r(t), \tag{38}$$

$$\lim_{t\to\infty} \lambda(t)a(t)\exp\left(-\int_0^t [(1-\tau^a)r(t) - n]ds\right) = 0. \tag{39}$$

Then, from (37) and (38),

$$-\frac{c(t)u''(c(t))}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} = (1-\tau^a)r(t) - \rho.$$

To focus on the effects of taxation, we assume that the government does not issue the public bond and $g(t) = 0$. Namely, the government's budget constraint is

$$\tau^a r(t)A(t) + T(t) = 0. \tag{40}$$

This means $T(t) < 0$. The capital income tax revenue is used for the redistribution to the households. Using the same procedure as in section 3, we obtain the dynamic system under capital income taxation:

$$\begin{aligned}
&\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t),\\
&-\frac{c(t)u''(c(t))}{c(t)}\frac{\dot{c}(t)}{c(t)} = (1-\tau^a)\Big[f'(k(t)) - \delta\Big] - \rho,\\
&\lim_{t\to\infty} k(t)\exp\left(-\int_0^t (f'(k(s)) - (n+\delta))ds\right) = 0.
\end{aligned}$$

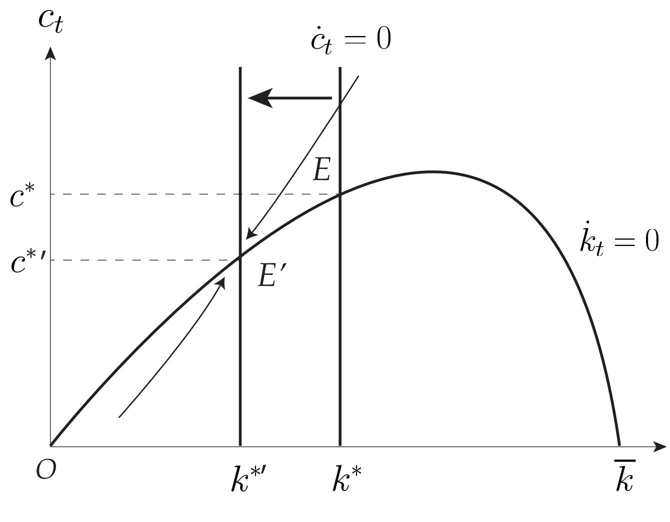

Figure 4: Effets of capital income taxation

Figure 4 shows how the capital income taxation affects the economy. The steady state moves from E to E':

1. the steady state capital stock falls from k^* to $k^{*'}$.
2. the steady state consumption also falls from c^* to $c^{*'}$.

However, note that during the transition, consumption increases temporarily.

6 Extension 2: Introducing Exogenous Technological Progress

In the baseline model, the equilibrium path of $(k(t), c(t))$ converges to the steady state (k^*, c^*). This means that the growth rate of all variables in per capita terms eventually becomes zero. To make the model more realistic, now we introduce the technological progress into the baseline model.

6.1 Labor-Augmenting Technological Progress

We extend the production function to

$$Y(t) = F(K(t), Z(t)L(t)),$$

Even if $K(t)$ or $L(t)$ does not change, $Y(t)$ increases if $Z(t)$ increases. Hereafter we interpret $Z(t)$ as the level of technology.

We introduce changes in $Z(t)$ to capture improvements in the technological know-how of the economy.

$$\dot{Z}(t)/Z(t) = \gamma > 0, \tag{41}$$

or equivalently

$$Z(t) = Z(0)\exp(\gamma t). \tag{42}$$

The technological progress such as (41) or (42) is called the *Labor-Augmenting Technological Progress.*

Define the following new variables:

$$\widetilde{y}(t) \equiv \frac{Y(t)}{Z(t)L(t)}, \quad \widetilde{k}(t) \equiv \frac{K(t)}{Z(t)L(t)}$$

We continue to assume F satisfies Assumptions 3–5, and define the function f as

$$f(\widetilde{k}) \equiv F(\widetilde{k}, 1)$$

The first–order–conditions of profit maximization problem are given by

$$R(t) = f'(\widetilde{k}(t)), \tag{43}$$

$$w(t) = [f(\widetilde{k}(t)) - \widetilde{k}(t)f'(\widetilde{k}(t))]Z(t), \tag{44}$$

which leads

$$R(t)\widetilde{k}(t) + w(t)/Z(t) = \widetilde{y}(t). \tag{45}$$

6.2 Balanced Growth Path

The conditions for utility maximization and the asset market equilibrium does not change from the baseline model. The dynamics of $\widetilde{k}(t)$ is given by

$$\begin{aligned} \dot{\widetilde{k}}(t)/\widetilde{k}(t) &= \dot{k}(t)/k(t) - \gamma \\ \rightarrow \quad \dot{\widetilde{k}}(t) &= f(\widetilde{k}(t)) - (n+\delta+\gamma)\widetilde{k}(t) - \widetilde{c}(t), \end{aligned} \tag{46}$$

where $\widetilde{c}(t) \equiv \frac{C(t)}{Z(t)L(t)}$. On the other hand, the dynamics of $\widetilde{c}(t)$ is given by

$$\begin{aligned} \frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} &= \frac{\dot{c}(t)}{c(t)} - \gamma \\ &= \left(-\frac{c(t)u''(c(t))}{u'(c(t))}\right)^{-1}(f'(\widetilde{k}(t)) - \delta - \rho) - \gamma \end{aligned}$$

Thus, $-cu''/u'$ must be constant.

Assumption 7. *$u(c)$ is specified as*

$$u(c) = \begin{cases} \dfrac{c^{1-\theta}-1}{1-\theta} & \textit{if } \theta > 0, \theta \neq 1, \\ \ln c & \textit{if } \theta = 1. \end{cases}$$

Note : $u(c)$ is called the *Constant Relative Risk Aversion (CRRA) utility* if it is specified as in the above assumption.

The dynamics of $\widetilde{c}(t)$ is eventually given by

$$\frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} = (1/\theta)(f'(\widetilde{k}(t)) - \delta - \rho - \theta\gamma). \tag{47}$$

Finally, the TVC is reduced to

$$\lim_{t\to\infty} \widetilde{k}(t) \exp\left(-\int_0^t (f'(\widetilde{k}_s) - n - \delta - \gamma)ds\right) = 0. \tag{48}$$

(46)–(48) jointly constitute the dynamic system. Since in steady state $\widetilde{k}(t)$ and $\widetilde{c}(t)$ must remain constant, from (46) and (47) we have

$$f'(\widetilde{k}^*) = \rho + \delta + \theta\gamma, \tag{49}$$

and

$$\widetilde{c}^* = f'(\widetilde{k}^*) - (n + \delta + \gamma)\widetilde{k}^*. \tag{50}$$

We can show the unique existence of $(\widetilde{k}^*, \widetilde{c}^*)$ which solves (49) and (50) in a way similar to the model without technological progress. The only additional condition in this case is that because there is growth, we have to make sure that the TVC is in fact satisfied. Substituting (49) into (48), we have

$$\lim_{t\to\infty} \widetilde{k}(t) \exp\{-[\rho - n - (1-\theta)\gamma]t\} = 0.$$

which can only hold if the following assumption is satisfied:

Assumption 8. $\rho - n > (1-\theta)\gamma$.

In the steady state, $\widetilde{k}(t)$ and $\widetilde{c}(t)$ are constant over time. From these definitions,

$$\dot{k}(t)/k(t) = \dot{c}(t)/c(t) = \gamma. \tag{51}$$

Furthermore, since $y(t) = \widetilde{y}(t)Z(t) = f(\widetilde{k}(t))Z(t)$, the growth rate of per capita GDP is γ in the long run.

→ In the steady state, all per capita variables grow at the rate of $\gamma > 0$.

→ In this model, the steady state is called the *Balanced Growth Path (BGP)*.

Proposition 6 (Balanced Growth Path)**.** *In steady state all per capita variable grow at the constant rate of technological progress,* $\gamma > 0$.

Summary

- The Ramsey–Cass–Koopmans model is based on the economic agents' dynamics optimization.
- There exists a unique steady state where both of physical capital and consumption are positive.
- Saddle point stability of the steady state means the uniqueness of competitive equilibrium path in this model.

- The competitive equilibrium path in this model corresponds to the social planner's optimal path which achieves the first-best allocation.
- In the baseline Ramsey model, the government spending crowds out private consumption.
- Capital income taxation harms the households' savings, thereby capital accumulation. In consequence, the capital stock in the steady state decreases if the tax rate becomes higher.
- By introducing the labor-augmenting technological change to the baseline model, all per capita variables (per capita GDP, capital, consumption...) become to grow at the same constant rate of technological progress in the long-run.

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