IEE. B402 Advanced Macroeconomics

A Firm's Intertemporal Behavior

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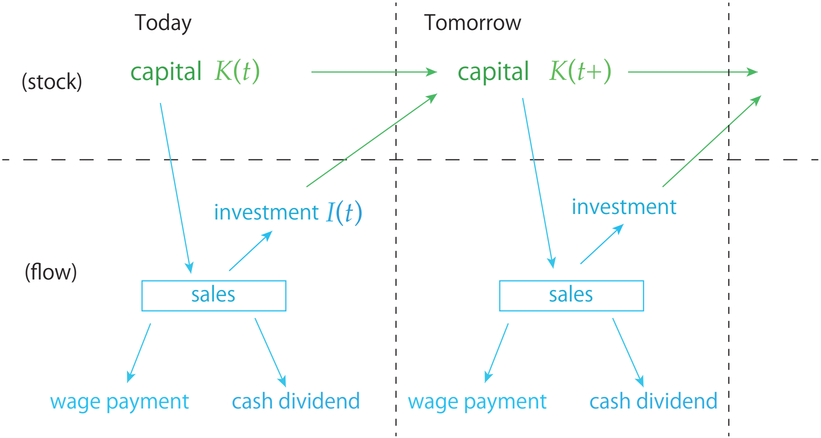
June 21, 2019 (revised)

Introduction

- Not only households, but also firms often face a situation of dynamic decision making:



Plan of Talk

1. The canonical model of a firm's intertemporal optimization for production and investment.
2. The q-theory of investment (an extenstion)
3. Economic implications

Notation

- $K(t)$: capital stock, or simply capital;
- $I(t)$: gross capital investment, or simply investment;
- $L(t)$: demand for labor;
- $Y(t)$: output;
- $r(t)$: interest rate;
- $w(t)$: wage rate;
- $F(K, L)$: production function of a firm;
- $V(0)$: stock value of a firm evaluated at the initial date;
- $\delta(\geq 0)$: capital depreciation rate (constant).

Setup

- There is a single final good, which is used for consumption and investment.
- The price of this good is normalized to unity (i.e., the good is taken as the numeraire).
- Consider a firm, of which output at date $t \geq 0$ is given by

$$Y(t) = F(K(t), L(t)). \tag{1}$$

- It pays the wage rate $w(t)$ for each unit of labor $L(t)$. In addition, it pays the cost of investment for production in the future.
- The change in the firm's capital stock is then given by

$$\dot{K}(t) = I(t) - \delta K(t). \tag{2}$$

Note: $I(t)$ can be negative.

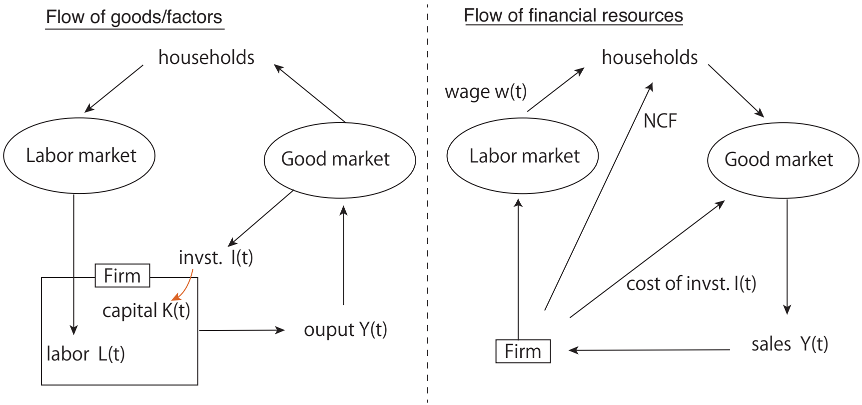
Setup (cont'd)

- Note that the cost of investment is equal to the cost of purchasing the good for the investment purpose.
  $\Rightarrow$ The firm pays $I(t)$ for an amount of investment $I(t)$.
  $(*)$ This assumption is relaxed later.
- Accordingly, the firm's net cash flow is given by

$$\text{Net cash flow} = F(K(t), L(t)) - w(t)L(t) - I(t). \tag{3}$$

- The net cash flow (NCF) given in (3) is payed out as dividends to the shareowners.
- The stock value of this firm, $V(0)$, is defined as the sum of present value of the NCF between 0 and ∞.

Schematic



Flow of goods/factors
households
Labor market
Good market
invst. I(t)
Firm
capital K(t)
labor L(t)
ouput Y(t)
Flow of financial resources
households
wage w(t)
NCF
Labor market
Good market
cost of invst. I(t)
Firm
sales Y(t)

A Firm's Stock Value Maximization Problem

- The canonical problem of a firm in a dynamic environment:

$$\begin{aligned} \max \quad & V(0) = \int_0^\infty e^{-\int_0^t r(s)ds} \Big[F(K(t), L(t)) - w(t)L(t) - I(t)\Big] dt \\ \text{s.t.} \quad & \dot{K}(t) = I(t) - \delta K(t), \\ & K(0) \text{ given.} \end{aligned} \tag{2}$$

- The firm chooses $(L(t), I(t), K(t))_{t \geq 0}$ so as to solve the above problem.
- We assume

$$F_K \equiv \partial F / \partial K > 0, \quad F_L \equiv \partial F / \partial L > 0,$$
$$F_{KK} \equiv \partial^2 F / (\partial K)^2 < 0, \quad F_{LL} \equiv \partial^2 F / (\partial L)^2 < 0.$$

Derivation of Conditions for Optimization

$\rightarrow$ Quiz in the class.

Step. 1

- Let $q(t)$ denote the Lagrange multiplier associated with (2).
- Construct the Lagrangian:

$$\mathcal{L} = \int_0^\infty e^{-\int_0^t r(s)ds} \Big[F(K(t), L(t)) - w(t)L(t) - I(t) \\ + q(t)(I(t) - \delta K(t) - \dot{K}(t)) \Big] dt.$$

- From the slides on Jun. 14, the current-value Hamiltonian is given by

$$H(t) = \boxed{},$$

Derivation of Conditions for Optimization

Step. 2

- Rewrite $\mathcal{L}$ as

$$\mathcal{L} = \int_0^\infty e^{-\int_0^t r(s)ds} H(t)dt - \int_0^\infty e^{-\int_0^t r(s)ds} q(t)\dot{K}(t)dt.$$

- Integrating the second term by parts,

$$\int_0^\infty e^{-\int_0^t r(s)ds} q(t)\dot{K}(t)dt = \boxed{\quad} - \boxed{\quad} \tag{4}$$

Derivation of Conditions for Optimization

Step. 3

- Then, the TVC means

$$\lim_{t\to\infty} e^{-\int_0^t r(s)ds} q(t)K(t) = 0. \tag{5}$$

- In addition, substituting (4) back into $\mathcal{L}$ on pp. 10, we obtain the conditions for optimization with respect to L, I, and K.

$$L(t): \quad \frac{\partial H}{\partial L(t)} = 0 \Leftrightarrow \boxed{}, \tag{6}$$

$$I(t): \quad \frac{\partial H}{\partial I(t)} = 0 \Leftrightarrow \boxed{}, \tag{7}$$

$$K(t): \quad \dot{q}(t) = r(t)q(t) - \frac{\partial H}{\partial K(t)}$$

$$\Leftrightarrow \boxed{}. \tag{8}$$

Dynamic vs Static Optimization

- In this simple model, the multiplier $q(t)$ is always equal to $\boxed{}$ from (7).
- Then, (8) implies

$$\boxed{}.$$

- Let $R(t)$ denote $R(t) = r(t) + \delta$. Then, the same conditions are obtained from the following static profit maximization problem:

$$\max_{K(t),L(t)} \; F(K(t), L(t)) - w(t)L(t) - R(t)K(t). \tag{9}$$

An Extension: The q-Theory of Investment

Modification

- We now assume that the cost of investment is given by

$$\text{Cost of investment} = I(t) + \Phi(I(t), K(t)), \tag{10}$$

where the second term is called the *adjustment cost of investment* (投資の調整費用).

- In the context of the investment theory, function Φ is often specified as

$$\Phi(I, K) = zI^2/K,$$

where $z > 0$ is an exogenously given parameter.

Problem

- The problem:

$$
\begin{aligned}
\max \quad & V(0) = \int_0^\infty e^{-\int_0^t r(s)ds} \Big[F(K(t), L(t)) - w(t)L(t) \\
& \qquad\qquad - I(t) - zI(t)^2/K(t) \Big] dt \\
\text{s.t.} \quad & \dot{K}(t) = I(t) - \delta K(t), \\
& K(0) \text{ given.}
\end{aligned}
$$

- The current-value Hamiltonian:

$$
H = F(K, L) - wL - I - zI^2/K + q(I - \delta K). \tag{11}
$$

Conditions for Optimization

- TVC is given by (5) also in this problem.
- On the other hand, the conditions for optimization with respect to L, I, and K are given by

$$L(t): \quad \frac{\partial H}{\partial L(t)} = 0 \Leftrightarrow \boxed{}, \tag{12}$$

$$I(t): \quad \frac{\partial H}{\partial I(t)} = 0 \Leftrightarrow \boxed{}, \tag{13}$$

$$\begin{aligned} K(t): \quad \dot{q}(t) &= r(t)q(t) - \frac{\partial H}{\partial K(t)} \\ &\Leftrightarrow \boxed{}. \end{aligned} \tag{14}$$

Economic Implications

- How can we evaluate the efficiency of firms' investment behavior in our real world?

  $\Downarrow$

- It is important to empirically check whether or not firms' investment satisfies (13), which gives the important theoretical prediction.
- In particular, from (13), we can obtain the following relationship:

$$I(t) \gtreqless 0 \Leftrightarrow q(t) \gtreqless \boxed{} \tag{15}$$

(Caution) Unfortunately, however, $q(t)$ is the Lagrange multiplier in the optimization problem. The data is not directly available.

Useful Theorem

Theorem

Suppose that the production function F is linearly homogenous:

$$F(xK, xL) = xF(K, L) \forall x \geq 0.$$

Then, the conditions for optimization of the problem (on pp. 15) given by (5) and (12)–(14) jointly imply $V(0) = q(0)K(0)$.

Proof.

Homework assignment □

- Then, in this case, we can obtain the data of $q(0)$ from the data of stock value $V(0)$ and capital $K(0)$.