IEE. B402 Advanced Macroeconomics

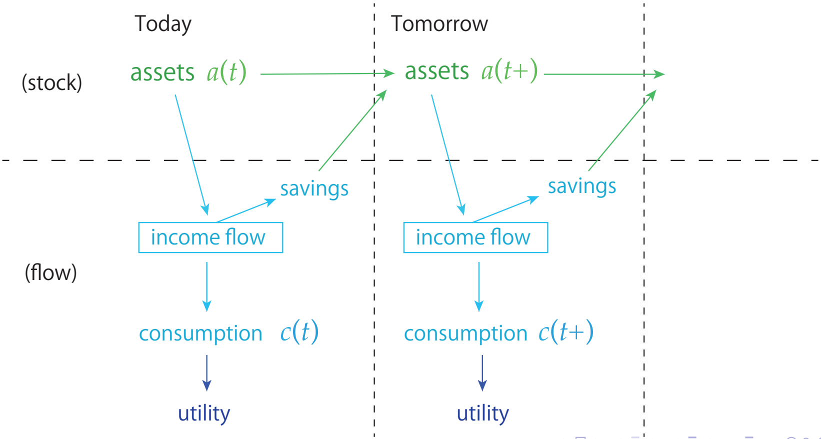
Introduction to Dynamic Optimization in Continuous Time

Ryoji Ohdoi
Dept. of Indust. Eng. & Econ, Tokyo Tech

June 14, 2019

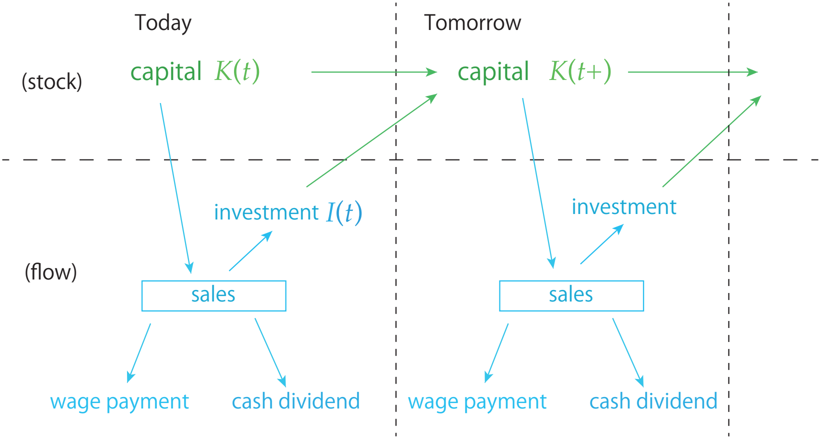
Introduction

- In many facets of life, we often face a situation of dynamic decision making.
- Example: our intertemporal allocation of consumption



Another Example

- Firms' intertemporal plan of production



Two Classes of Variables

- We start with a classification of variables.
- In a dynamic environment, variables are divided into two classes, *state variables* (状態変数) and *control variables* (制御変数).
- The movement of state variables is governed by first order differential equations, called the *transition equation* (推移方程式).
- Time is continuous here and indexed by t. Let $x(t)$ denote a state variable and $u(t)$ denote a control variable.
- Then, the transition equation is generally expressed as

$$\dot{x}(t)\left(\equiv \frac{dx(t)}{dt}\right) = G(x(t), u(t), t). \tag{1}$$

  ($*$) Hereafter, a dot over a variable indicates its time derivative.
- G is the function which governs the movement of $x(t)$, adequately specified depending on the context (see the next slide).

Examples

1. In a household's behavior,
   - Her assets corresponds to the state variable $(x(t) = a(t))$, while consumption corresponds to the control variables $(u(t) = c(t))$.
   - Her flow budget constraint corresponds to the transition equation:

     $$\dot{a}(t) = r(t)a(t) + w(t) - c(t),$$

     where $r(t)$ is the interest rate and $w(t)$ is the wage at time t.
2. In a firm's behavior,
   - Its capital corresponds to the state variable $(x(t) = K(t))$, while investment and labor demand correspond to the control variables $(u(t) = (I(t), L(t)))$.
   - Dynamic equation of capital corresponds to the transition equation:

     $$\dot{K}(t) = I(t) - \delta K(t),$$

     where $\delta \geq 0$ is the rate of capital depreciation.

Canonical Problem

- For simplicity, we assume that both of x and u are one-dimensional.
- Then, the simplest continuous-time optimization problem is written as

$$\max_{(x(t),u(t))_{t\geq 0}} J = \int_0^T F(x(t),u(t),t)dt$$

$$\text{s.t.} \quad \dot{x}(t) = G(x(t),u(t),t), \tag{1}$$

$$x(0) = x_0 \text{ given}, \tag{2}$$

  where we refer to J and F respectively as the objective function and the instantaneous return function.
- For the moment, we assume that the end point T is finite.
- The constraint (2) means that the initial value of the state variable is historically given.

Additional Constraint

- Additional constraint:

  In addition to (1) and (2), the optimization problem in economics often includes the following inequality constraint at the terminal date:

$$b(T)x(T) \geq 0, \tag{3}$$

  where $b(T)$ is the variable dependent on T, the example of which is the discount factor applied to $x(T)$.

Example: A Household's Utility Maximization

- A household's dynamic utility maximization problem:

$$\max_{(a(t),c(t))_{t\geq 0}} U = \int_0^T e^{-\rho t} \ln c(t) dt$$

$$\text{s.t.} \quad \dot{a}(t) = r(t)a(t) + w(t) - c(t),$$

$$a(0) = a_0 \text{ given},$$

$$e^{-\int_0^T r(s)ds} a(T) \geq 0,$$

where

- $\rho > 0$ is the *subjective discount rate* (主観的割引率), applied to the utility from consumption.
  $\rightarrow e^{-\rho t} \ln c(t) < \ln c(t)$ $(\forall t > 0)$ means that at the initial date, she discounts her future utilities.

- On the other hand, $e^{-\int_0^T r(s)ds}$ captures the time discounting about assets at the terminal date, $a(T)$.

Assumption

- Consider the general problem on pp. 6.
- Assumption:

  F and G are continuously differentiable functions of three independent arguments.

Lagrangian and Hamiltonian

- Set up the Lagrangian:

$$\mathcal{L} = \int_0^T \Big\{ F(x(t), u(t), t) + \mu(t) \Big[G(x(t), u(t), t) - \dot{x}(t) \Big] \Big\}\, dt \\ + \eta(T) b(T) x(T),$$

where $\mu(t)$ is the Lagrangian multiplier associated with the constraint (1), and ν is the multiplier with the constraint (3).

- Let us define the function H:

$$H(x(t), u(t), \mu(t), t) = F(x(t), u(t), t) + \mu(t) G(x(t), u(t), t).$$

H is called the *Hamiltonian*.

Reduced Form of $\mathcal{L}$

- The Lagrangian is rewritten as

$$\mathcal{L} = \int_0^T H(x(t), u(t), \mu(t), t)dt - \int_0^T \mu(t)\dot{x}(t)dt + \nu b(T)x(T).$$

- Integrating the second term by parts, we can obtain

$$\int_0^T \mu(t)\dot{x}(t)dt = \mu(T)x(T) - \mu(0)x_0 - \int_0^T x(t)\dot{\mu}(t)dt. \quad (4)$$

- Using (4), the Lagrangian is eventually reduced to

$$\mathcal{L} = \int_0^T \Big[H(x(t), u(t), \mu(t), t) + x(t)\dot{\mu}(t)\Big]dt$$
$$- \mu(T)x(T) + \nu b(T)x(T) + \mu(0)x_0. \quad (5)$$

First-Order Conditions

- Then, in analogy with solving a nonlinear programming problem, we obtain the first-order conditions (hereafter F.O.Cs) with respect to $u(t)$ and $x(t)$:

$$u(t): \quad \frac{\partial H(\cdot)}{\partial u(t)} = 0 \Leftrightarrow \frac{\partial F}{\partial u(t)} + \mu(t)\frac{\partial G}{\partial u(t)} = 0, \tag{6}$$

$$x(t): \quad \frac{\partial H(\cdot)}{\partial x(t)} + \dot{\mu}(t) = 0 \Leftrightarrow \dot{\mu}(t) = -\left[\frac{\partial F}{\partial x(t)} + \mu(t)\frac{\partial G}{\partial x(t)}\right]. \tag{7}$$

  Equation (7) is called the Euler–Lagrange equation. In economics, it is simply called the *Euler equation*.

- F.O.Cs with respect to $\mu(t)$ $\rightarrow$ the transition equation (1).

Transversality Condition

- Note that equation (7) is the condition for $t \in [0, T)$. The condition with respect to $x(T)$ is given by

$$\mu(T) = \nu b(T).$$

- Moreover, the (Kuhn-Tucker) multiplier ν must satisfy the following complementary slackness condition:

$$\nu b(T) x(T) = 0.$$

- The above two conditions jointly imply

$$\mu(T) x(T) = 0. \tag{8}$$

Condition (8) is called the *transversality condition* (横断性条件), often abbreviated as the TVC.

On the Sufficiency

- Note that conditions (6)–(8) are the *necessary* conditions for optimization.
- So it is still silent whether or not these conditions are *sufficient* for that.

$\Downarrow$

Theorem (Sufficiency)

If the Hamiltonian H is concave with respect to (x(t), u(t)), then conditions (6) – (8) are also the sufficient conditions for optimization.

Proof.

Appendix. □

Application: A Household's Utility Maximization

- Now consider the utility maximization problem on pp. 8.
- The Hamiltonian is given by

$$H(a, c, \mu, t) = e^{-\rho t} \ln c(t) + \mu(t) \left[r(t)a(t) + w(t) - c(t) \right].$$

  Note that H is concave with respect to (a, c).

- The conditions for utility maximization:

$$c(t): \quad \frac{\partial H}{\partial c(t)} = 0 \Leftrightarrow \boxed{\qquad\qquad\qquad} \tag{9}$$

$$a(t): \quad \dot{\mu}(t) + \frac{\partial H}{\partial a(t)} = 0 \Leftrightarrow \boxed{\qquad\qquad\qquad} \tag{10}$$

$$\text{TVC}: \quad \mu(T)a(T) = 0 \Leftrightarrow \boxed{\qquad\qquad\qquad} \tag{11}$$

  (9) and (11) jointly imply $\boxed{\qquad\qquad}$.

- Economic implications of the dynamic utility maximization are discussed in an upcoming class.

The Optimization Problem with Time Discounting

- Economically interesting problems often specify the instantaneous return function F as

$$F(x(t), u(t), t) = e^{-\rho t} f(x(t), u(t), t),$$

which means that one unit of return at time t is evaluated as $e^{-\rho t} < 1$ at the initial time.

- The problem is now given by

$$\begin{aligned} \max_{(x(t),u(t))_{t\geq 0}} J = \int_0^T & e^{-\rho t} f(x(t), u(t), t) dt \\ \text{s.t.} \quad & \dot{x}(t) = G(x(t), u(t), t), \\ & x(0) = x_0 \text{ given}, \\ & b(T)x(T) \geq 0. \end{aligned}$$

Two Types of Hamiltonian

- The Hamiltonian takes the form:

$$H(x(t), u(t), \mu(t), t) = e^{-\rho t} f(x(t), u(t), t) + \mu(t) G(x(t), u(t), t). \tag{12}$$

  In the discounted optimization problem, function (12) is called the *present-value Hamiltonian* (現在価値ハミルトニアン).

- When we introduce the new variable, $\lambda(t) \equiv \mu(t) e^{\rho t}$, we can rewrite (12) as

$$H(x(t), u(t), \mu(t), t) = e^{-\rho t} \hat{H}(x(t), u(t), \lambda(t), t),$$

  where

$$\hat{H}(x(t), u(t), \lambda(t), t) = f(x(t), u(t), t) + \lambda(t) G(x(t), u(t), t). \tag{13}$$

  (13) is the *current-value Hamiltonian* (当該価値ハミルトニアン).

F.O.Cs

- Needless to say, whichever we use, (12) or (13), we can obtain the same F.O.Cs.
- When we use the current-value Hamiltonian (13), we can obtain the following conditions:

$$u(t): \quad \frac{\partial \hat{H}}{\partial u(t)} = 0 \Leftrightarrow \frac{\partial f}{\partial u(t)} + \lambda(t)\frac{\partial G}{\partial u(t)} = 0, \tag{14}$$

$$x(t): \quad \frac{\partial \hat{H}}{\partial x(t)} - \rho\lambda(t) + \dot{\lambda}(t) = 0$$

$$\Leftrightarrow \dot{\lambda}(t) = \rho\lambda(t) - \left[\frac{\partial f}{\partial x(t)} + \lambda(t)\frac{\partial G}{\partial x(t)}\right], \tag{15}$$

$$\text{TVC}: \quad e^{-\rho T}\lambda(T)x(T) = 0. \tag{16}$$

- Once we recall the definition of $\lambda(t)$: $\lambda(t) \equiv e^{\rho t}\mu(t)$, we can easily verify that the conditions (14)–(16) are equivalent to (6)–(8).

A Household's Utility Maximization (Once Again)

- Consider the utility maximization problem on pp. 8 again.
- The *current-value* Hamiltonian is given by

$$\hat{H}(a, c, \mu, t) = \ln c(t) + \lambda(t) \left[r(t)a(t) + w(t) - c(t) \right].$$

- The conditions for utility maximization:

$$c(t): \quad \frac{\partial \hat{H}}{\partial c(t)} = 0 \Leftrightarrow \boxed{} \tag{17}$$

$$a(t): \quad \dot{\lambda}(t) - \rho\lambda(t) + \frac{\partial \hat{H}}{\partial a(t)} = 0 \Leftrightarrow \boxed{} \tag{18}$$

$$\text{TVC}: \quad e^{-\rho T}\lambda(T)a(T) = 0 \Leftrightarrow \boxed{} \tag{19}$$

Infinite-Horizon Problem

- Most economic models, not only in macroeconomics, but also in repeated games, political economy and so on, are formulated as infinite-horizon problems.

- Consider the following problem by taking a limit of $T \to \infty$ in the original problem:

$$
\begin{aligned}
\max_{(x(t),u(t))_{t\geq 0}} \quad & J = \int_0^\infty e^{-\rho t} f(x(t), u(t), t)dt \\
\text{s.t.} \quad & \dot{x}(t) = G(x(t), u(t), t), \\
& x(0) = x_0 \text{ given}, \\
& \lim_{T\to\infty} b(T)x(T) \geq 0.
\end{aligned}
$$

Infinite-Horizon Problem

- (6)–(7) or (14)–(15) provide the conditions for optimization also in this infinite-horizon problem.
- On the other hand, the TVC (8) or (16) is now replaced by

$$\lim_{T \to \infty} \mu(T) x(T) = 0, \tag{20}$$

or

$$\lim_{T \to \infty} e^{-\rho T} \lambda(T) x(T) = 0. \tag{21}$$