Ramsey–Cass–Koopmans Model (2)
IEE.B402. Advanced Macroeconomics

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Plan

- The role of policy
  - Government spending
  - Debt financing
- Exogenous technological progress
  - Setup
  - Balanced growth path (BGP)

Government Spending

- Now suppose that the government consumes $G(t)$ units of the final good.
  $\rightarrow$ In per capita terms, $g(t) = G(t)/L(t)$.
- The government levies lump-sum taxes, $T(t)$ to finance the expenditure.
- Therefore the government's budget constraint is

$$T(t)/L(t) = g(t). \tag{30}$$

  ($*$) This situation is called balanced budget.
- We assume that $(g(t)$ is exogenously given.
  $\rightarrow$ Given the path of $g(t)$, (30) determines the path of $T(t)$.

Households

- A representative household's problem:

$$\max_{(a(t),c(t))_{t\geq 0}} \quad U = \int_0^\infty e^{-(\rho-n)t} u(c(t))dt$$

$$\text{s.t.} \quad \dot{a}(t) = (r(t)-n)a(t) + w(t) - c(t) - T(t)/L(t), \tag{31}$$

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t (r(s)-n)ds\right) \geq 0,$$

$$a(0) \quad \text{given.} \tag{32}$$

Quiz

Show that the Euler equation (8) does not change.

Equilibrium

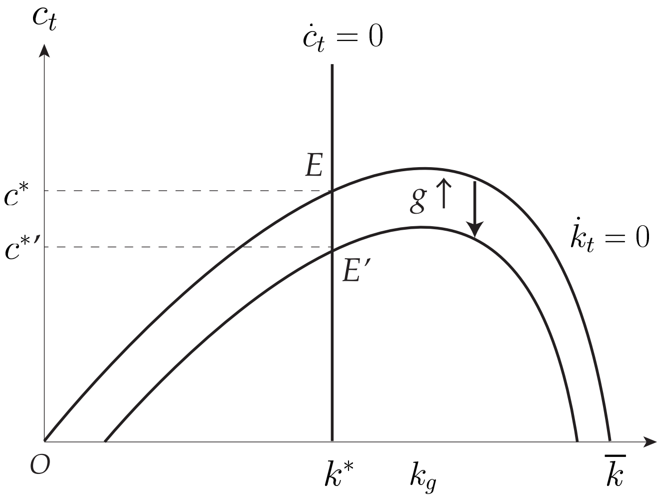
- Asset market equilibrium: $a(t) = k(t)$ does not change.

  $\Downarrow$

- Dynamic system: equations (17) and (18) in Section 3 still hold, whereas (16) is replaced by

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t) - g(t)$$

Effect of gov. spending on the steady state



Debt Financing

- Now relax the balanced-budget assumption (30) so that the government is now allowed to borrow by issuing debts.
- Let $B(t) \geq 0$ denote the stock of government debt at date t.

  $\downarrow$
- The government's budget constraint (30) is replaced by

$$\underbrace{T(t) + \dot{B}(t)}_{\text{Revenue}} = \underbrace{r(t)B(t) + G(t)}_{\text{Expenditure}}$$

or equivalently,

$$\dot{B}(t) = \underbrace{r(t)B(t) + \underbrace{G(t) - T(t)}_{\text{Primary deficit}}}_{\text{Budget deficit}} \tag{A1}$$

Equilibrium

- Neither the households' nor firms' behavior changes at all.
  - Euler eq. is (8)
  - Firms' F.O.Cs are (11) and (12)

$\downarrow$

The dynamics of consumption is given by (17) also in this case.

- The asset market equilibrium is now given by $a(t) = k(t) + b(t)$.
  - $b(t) \equiv B(t)/L(t)$: the per-captia amount of debts.

- From (A1), we can obtain the government budget constraint in per-capita terms:

$$\dot{b}(t) = (r(t) - n)b(t) + g(t) - T(t)/L(t) \qquad \text{(A2)}$$

Equilibrium

- Substituting (A2) and $a(t) = k(t) + b(t)$ into the household budget constraint (31), we obtain the dynamics of capital as follows:

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t) - g(t)$$

$\downarrow$

Proposition 1

For a given time path of the government spending $G(t)$ (or $g(t)$), financing this spending plan through distortionless taxation and budget deficit are indifferent.

This property is called the Ricardian Neutrality or Ricardian Equivalence.

The Ramsey–Cass–Koopmans Model with Exogenous Technological Progress

Introduction of Technological Progress

- We extend the production function to the following:

$$Y(t) = F(K(t), Z(t)L(t))$$

  $Z(t)$: technology level at time t

- The technology level grows at the exogenous rate of $\gamma > 0$:

$$\dot{Z}(t)/Z(t) = \gamma > 0, \tag{41}$$

  or equivalently

$$Z(t) = Z(0)\exp(\gamma t). \tag{42}$$

  The technological progress such as (41) or (42) is called the Labor-Augmenting Technological Progress.

- We continue to assume F satisfies Assumptions 3–5.

Firms' Behavior

- Define the following new variables:

$$\widetilde{y}(t) \equiv \frac{Y(t)}{Z(t)L(t)}, \quad \widetilde{k}(t) \equiv \frac{K(t)}{Z(t)L(t)} = \frac{k(t)}{Z(t)}$$

and define the function f as

$$f(\widetilde{k}) \equiv F(\widetilde{k}, 1)$$

$\downarrow$

- Profit maximization problem of a representative firm:

$$\max_{\widetilde{k}, L} \left[f(\widetilde{k}(t)) - R(t)\widetilde{k}(t) - w(t) \right] Z(t)L(t)$$

The first-order-conditions (F.O.Cs) are given by

$$R(t) = f'(\widetilde{k}(t)) \tag{43}$$

$$w(t) = [f(\widetilde{k}(t)) - \widetilde{k}(t) f'(\widetilde{k}(t))] Z(t) \tag{44}$$

Households' Behavior

- Households' behavior does not change from Section 2.1.
  - Flow budget constraint in per-capita is (4)
  - Euler equation is (8)

$\downarrow$

- Hereafter, specify the instantaneous utility function u as the following CRRA form:

$$u(c) = \begin{cases} \dfrac{c^{1-\theta} - 1}{1-\theta} & \text{if } \theta > 0, \theta \neq 1, \\ \ln c & \text{if } \theta = 1. \end{cases}$$

$\downarrow$

- Euler equation (8) becomes

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho) \tag{A3}$$

Equilibrium

- Asset market equilibrium is the same as Section 2, which is given by (14): $a(t) = k(t) = \widetilde{k}(t)Z(t)$.
- From this equation and the household budget (4), we obtain the dynamics of $\widetilde{k}(t)$ as follows:

$$\dot{\widetilde{k}}(t) = f(\widetilde{k}(t)) - (n + \delta + \gamma)\widetilde{k}(t) - \widetilde{c}(t) \quad (46)$$

where $\widetilde{c}(t) = c(t)/Z(t)$

Quiz

Show it.

Equilibrium

- Using (41), (43) and (A3), the dynamics of $\widetilde{c}(t)$ is given by

$$\frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} = \frac{\dot{c}(t)}{c(t)} - \gamma$$
$$= \frac{1}{\theta}\left(f'(\widetilde{k}(t)) - \delta - \rho - \theta\gamma\right) \tag{47}$$
$$(\because R(t) - \delta = r(t))$$

- Finally, TVC is given by

$$\lim_{t\to\infty} \widetilde{k}(t) \exp\left(-\int_0^t (f'(\widetilde{k}_s) - n - \delta - \gamma)ds\right) = 0 \tag{48}$$

(46)–(48) jointly constitute the autonomous dynamic system.

Balanced Growth Path

- The existence, uniqueness, and stability of steady state of the system (46)–(48) is guaranteed in the same manner as Sections 3.2 and 3.3.
  $\downarrow$
- In the long run, $(\widetilde{k}(t), \widetilde{c}(t))$ converges to $(\widetilde{k}^*, \widetilde{c}^*)$, where they are given by

$$f'(\widetilde{k}^*) = \rho + \delta + \theta\gamma \tag{49}$$

$$\widetilde{c}^* = f(\widetilde{k}^*) - (n + \delta + \gamma)\widetilde{k}^* \tag{50}$$

Balanced Growth Path

- Thus, $\widetilde{k}(t)$ and $\widetilde{c}(t)$ eventually become constant over time.
  From these definitions,

$$\dot{k}(t)/k(t) = \dot{c}(t)/c(t) = \gamma \tag{51}$$

- Furthermore, since the per-capita GDP is given by

$$y(t) = f(\widetilde{k}(t))Z(t)$$

  its growth rate is also give by γ in the long run.
- In the steady state, all per capita variables grow at the rate of $\gamma > 0$.
  This is called the Balanced Growth Path (BGP).

Proposition 2 (Balanced Growth Path)

In steady state all per capita variable grow at the constant rate of technological progress, $\gamma > 0$.