IEE. B402 Advanced Macroeconomics

A Household's Intertemporal Behavior

Ryoji Ohdoi
Dept. of Indust. Eng. & Econ, Tokyo Tech

June 18, 2019

Notation

- U: the household's lifetime utility;
- $u(c)$: the *instantaneous utility function* (瞬時効用関数), assumed to be $u' > 0$ and $u'' < 0$;
- $\rho > 0$: the subjective discount rate, applied to the $u(c)$;
- $a(t)$: her assets (state variable);
- $c(t)$: her consumption (control variable);
- $r(t)$: the interest rate;
- $w(t)$: the wage rate.

A Household's Utility Maximization

- The canonical problem of a household's utility maximization in a dynamic environment:

$$\max_{(c(t)),a(t))_{t\geq 0}} \quad U = \int_0^\infty e^{-\rho t} u(c(t))dt$$

$$\text{s.t.} \quad \dot{a}(t) = r(t)a(t) + w(t) - c(t), \tag{1}$$

$$a(0) = a_0 \text{ given}, \tag{2}$$

$$\lim_{t\to\infty} e^{-\int_0^t r(s)ds} a(t) \geq 0, \tag{3}$$

- The constraint (1) is the budget constraint, (2) is the initial condition for her assets, and (3) is called the *No-Ponzi game condition (ポンジー・ゲーム禁止条件)* abbreviated as the NPG.
- Without the NPG, the household can increase her consumption by borrowing to such level that feasibility is violated.

Conditions for Optimization

- The current-value Hamiltonian:

$$H = u(c(t)) + \lambda(t)\Big[r(t)a(t) + w(t) - c(t)\Big],$$

where $\lambda(t)$ is the multiplier. it is also called the *costate variable* (共役変数).

- Conditions for Optimization:

$$\frac{\partial H}{\partial c(t)} = 0 \Leftrightarrow \boxed{}, \tag{4}$$

$$\dot{\lambda} = \rho\lambda(t) - \frac{\partial H}{\partial a(t)} \Leftrightarrow \boxed{}, \tag{5}$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)a(t) = 0. \tag{6}$$

Euler Equation and TVC

- Using (4), the Euler equation (5) is rewritten as

$$-\frac{c(t)u''(c(t))}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} = r(t) - \rho. \tag{7}$$

Proof.

Homework assignment □

- Using (5), the TVC (6) is rewritten as

$$\lim_{t\to\infty} e^{-\int_0^t r(s)ds} a(t) = 0. \tag{8}$$

Proof.

Homework assignment □

Economic Implication (i): What does the Euler Equation mean?

Economic Implications of Euler Equation

- What does the Euler equation provide us?
- Suppose that the household decreases c_t but increases $c_{t+\Delta t}$ with U unchanged.
- By differentiating the life-time utility and imposing $dU = 0$, we have the marginal rate of substitution (MRS) of consumption at t for $t + \Delta t$:

$$
\begin{aligned}
dU = 0 &\Rightarrow u'(c_t)dc_t + e^{-\rho \Delta t}u'(c_{t+\Delta t})dc_{t+\Delta t} \\
&\Rightarrow -\frac{dc_{t+\Delta t}}{dc_t} = \frac{u'(c_t)}{e^{-\rho \Delta t}u'(c_{t+\Delta t})}.
\end{aligned}
$$

Economic Implications of Euler Equation (cont'd)

- In analogy with a two-period utility maximization problem, the above MRS must be equal to the gross interest rate:

$$\frac{u'(c_t)}{e^{-\rho\Delta t}u'(c_{t+\Delta t})} = 1 + r_t\Delta t$$

$$\Rightarrow \quad \frac{1}{\Delta t}\frac{u'(c_t) - e^{-\rho\Delta t}u'(c_{t+\Delta t})}{e^{-\rho\Delta t}u'(c_{t+\Delta t})} = r_t.$$

- Taking the limit $\Delta t \to 0$, we obtain

$$\rho - \frac{c_t u''(c_t)}{u'(c_t)}\frac{\dot{c}_t}{c_t} = r_t.$$

  This equation is exactly the Euler equation (7).

- Therefore, the Euler equation (7) means "MRS=rate of return from saving."

Economic Implications (ii): How does consumption change over time?

How Does Consumption Change Over Time?

- Euler equation (7) implies that consumption increases or decreases over time depending on whether the interest rate exceeds or is less than the subjective discount rate.

$$\dot{c}_t \gtreqless 0 \;\Leftrightarrow\; r \gtreqless \rho,$$

  which comes from the fact that $-cu''/u' > 0$ as long as $c > 0$.
- Therefore, the sign of gap $r - \rho$ determines whether or not consumption grows over time.

How Does Consumption Change Over Time?

- On the other hand, $-cu''/u' > 0$, which is the elasticity of marginal utility, determines steepness of the slope of consumption.

$$\frac{\dot{c}_t}{c_t} = \left(-\frac{c_t u''(c_t)}{u'(c_t)}\right)^{-1} (r - \rho).$$

$\rightarrow (-cu''/u')^{-1}$ is therefore called the degree of *intertemporal elasticity of substitution (IES, 異時点間の代替の弾力性)*.

Useful Specification

- The following specification about the instantaneous utility function u is frequently used in intertemporal optimizing models:

$$u(c) = \begin{cases} \dfrac{c^{1-\theta} - 1}{1 - \theta} & \text{for } \theta > 0, \theta \neq 1, \\ \ln c & \text{for } \theta = 1. \end{cases} \tag{9}$$

- This function is called the *CRRA utility function*, where CRRA is the abbreviated name of the "Constant Relative Risk Aversion."

- Under the specification (9), the Euler equation (7) is given by

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}. \tag{10}$$

Proof.

Quiz □

Economic Implication (iii): Utility-Maximizing Consumption Level

How Consumption is Determined

- Then, how does the household decide the optimal paths of consumption and assets?
- For simplicity, hereafter, we assume
  1. The instantaneous utility function u is specified as the CRRA form given in (9);
  2. The interest rate is constant over time, $r(t) = r > \rho$ for all t;
  3. the wage rate $w(t)$ grows at the constant rate of $g \in [0, r)$:

$$\dot{w}(t) = gw(t) \Leftrightarrow w(t) = w(0)e^{gt}. \quad (11)$$

How Consumption is Determined

- At first, from these assumptions, the Euler equation (7) is rewritten as
$$\frac{\dot{c}(t)}{c(t)} = \gamma.$$
where $\gamma \equiv (r - \rho)/\theta$.
- Then, we have
$$c(t) = c(0)e^{\gamma t} \tag{12}$$

Intertemporal Budget Constraint

- From the budget constraint (1) and the TVC (6), we obtain the following equation:

$$\int_0^\infty c(t)e^{-rt}dt = a(0) + \int_0^\infty w(t)e^{-rt}dt, \tag{13}$$

  the derivation of which is given at the class.

- Equation (13) is called the *intertemporal budget constraint*:
  - Left-had side is the lifetime consumption, the sum of discounted value of consumption
  - Right-had side is the lifetime income, which is in turn classified into
    1. the financial wealth in the initial date $a(0)$;
    2. the human wealth, defined as the sum of discounted value of wage income.

Utility-Maximizing Consumption Level

Assumption

$(1-\theta)r < \rho.$

$\Downarrow$

- Substituting (11) and (12) into (13) yields

$$c(0) = \frac{\rho - (1-\theta)r}{\theta}\left(a(0) + \frac{w(0)}{r-g}\right),$$

  the derivation of which is given at the class.

- Since the initial date can be arbitrarily chosen, we have

$$c(t) = \frac{\rho - (1-\theta)r}{\theta}\left(a(t) + \frac{w(t)}{r-g}\right) \forall t. \tag{14}$$

Expectation about Future $\rightarrow$ Current Reaction

- (14) provides the optimizing consumption level under a dynamic situation.
- It clearly shows that the decision maker immediately reacts to the change of her income not only at present, but also in the future.
- For example, suppose that the growth rate of wage goes up (that is, the increase in g).
- This does not mean the rise in her current wage $w(t)$, but means the increase in her future wage.
- (14) states that the household increase her consumption $c(t)$ in response to such a shock.
- This is because in a dynamic environment, her consumption is determined not from her current income, but from her lifetime income.