QIP Course 6: Matrix Expression of Quantum States

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Answers of prev. exercises

1–3. Omitted.
4. Prove that the measurement outcome is k **if** the state **before measurement** is $|\varphi_k\rangle$ in page ?.
Answer: Recall that the probability of getting outcome j is

$$\| |\varphi_j\rangle\langle\varphi_j| |\varphi_k\rangle \|^2 = \delta_{jk}.$$

Therefore the measurement outcome must be k under the given assumption.
Caution: **Many students (in Japan) somehow computed $A|\varphi\rangle$. But $A|\varphi\rangle$ has no physical meaning in this context. Such an answer was evaluated as incorrect.**
5. Prove that the state **before measurement** is $|\varphi_k\rangle$ **if** the measurement outcome is k in page 5-10.
Answer: It is enough to prove the contraposition: If the state before measurement is not $|\varphi_k\rangle$ then the measurement outcome is not k. We see that from the answer to Q4 that the contraposition obviously holds.

Overview of this unit

- density matrix
- privacy of superdense coding (next unit)

The density matrix is another representation for quantum states of physical objects.
The privacy means that nobody can steal any information from the transmitted qubit of the superdense coding.
Don't you think such proof may be difficult?
The use of density matrix seems the easiest way to prove it.

Properties of the trace of a matrix

$|\varphi_1\rangle$, ..., $|\varphi_n\rangle$: orthonormal basis.

$$\mathrm{Tr}A = \langle\varphi_1|A|\varphi_1\rangle + \cdots + \langle\varphi_n|A|\varphi_n\rangle.\text{(Definition)} \tag{1}$$

The value of the trace does not depend on the choice of ONB $\{|\varphi_1\rangle, ..., |\varphi_n\rangle\}$, i.e., two different ONBs give the same value of the trace. See your linear algebra textbook for these facts.

$$\mathrm{Tr}[\alpha A] = \alpha \mathrm{Tr}A.$$

$$\mathrm{Tr}[AB] = \mathrm{Tr}[BA]. \tag{2}$$

$$\mathrm{Tr}[A \otimes B] = \mathrm{Tr}A \cdot \mathrm{Tr}B. \tag{3}$$

Density matrix 1

Suppose that we do not completely know the state of a system, and that we know the state is $|\varphi_i\rangle$ with probability p_i.

Suppose also that we measure an observable

$$A = \sum_{k=1}^{n} k|\psi_k\rangle\langle\psi_k|,$$

where all eigenspaces are of dimension 1.

The probability of getting the measurement outcome k is

$$\begin{aligned}
\Pr[\text{outcome} = k] &= \sum_{i=1}^{n} \Pr[\text{outcome} = k \text{ and state} = |\varphi_i\rangle] \\
&= \sum_{i=1}^{n} \underbrace{\Pr[\text{state} = |\varphi_i\rangle]}_{=p_i} \underbrace{\Pr[\text{outcome} = k|\text{state} = |\varphi_i\rangle]}_{=\| |\psi_k\rangle\langle\psi_k| |\varphi_i\rangle \|^2}
\end{aligned}$$

$$
\begin{aligned}
\Pr[\text{outcome} = k] &= \sum_{i=1}^{n} p_i \| |\psi_k\rangle\langle\psi_k| |\varphi_i\rangle \|^2 \\
&= \sum_{i=1}^{n} p_i \langle\varphi_i| |\psi_k\rangle\langle\psi_k| |\psi_k\rangle\langle\psi_k| |\varphi_i\rangle \\
&= \sum_{i=1}^{n} p_i \langle\varphi_i| |\psi_k\rangle\langle\psi_k| |\varphi_i\rangle \\
&= \sum_{i=1}^{n} p_i \mathrm{Tr}[|\varphi_i\rangle\langle\varphi_i| |\psi_k\rangle\langle\psi_k|] \qquad (4) \\
&= \mathrm{Tr}\left[\underbrace{\left(\sum_{i=1}^{n} p_i |\varphi_i\rangle\langle\varphi_i| \right)}_{=\rho \text{ in p. 10}} |\psi_k\rangle\langle\psi_k| \right] \qquad (5)
\end{aligned}
$$

Explanation of Eq. (4)

Let $\{|\varphi_i\rangle, |u_2\rangle, ..., |u_n\rangle\}$ be an ONB. We will use this ONB for computation of the trace below.
Then by the definition of trace

$$
\begin{aligned}
& \mathrm{Tr}[|\varphi_i\rangle\langle\varphi_i||\psi_k\rangle\langle\psi_k|] && (6)\\
= \;& \langle\varphi_i||\varphi_i\rangle\langle\varphi_i||\psi_k\rangle\langle\psi_k||\varphi_i\rangle + \sum_{j=2}^{n}\langle u_j||\varphi_i\rangle\langle\varphi_i||\psi_k\rangle\langle\psi_k||u_j\rangle \\
= \;& \langle\varphi_i||\varphi_i\rangle\langle\varphi_i||\psi_k\rangle\langle\psi_k||\varphi_i\rangle && (7)\\
= \;& \langle\varphi_i||\psi_k\rangle\langle\psi_k||\varphi_i\rangle && (8)
\end{aligned}
$$

Density matrix 2

The previously presented state of the system can be represented by a density matrix

$$\rho = \sum_{i=1}^{n} p_i |\varphi_i\rangle\langle\varphi_i|,$$

and the probability of getting the measurement outcome k is given by $\mathrm{Tr}[\rho|\psi_k\rangle\langle\psi_k|]$ as Eq. (5).

If we apply an unitary matrix U on a system, then the state of system is changed from ρ to $U\rho U^*$.

If the system 1 is in state ρ_1, the system 2 is in state ρ_2, and systems 1 and 2 are NOT entangled with each other, then the state of systems 1 and 2 is $\rho_1 \otimes \rho_2$.

Measurement of a subsystem

Suppose that physical systems A and B are in the state

$$\tau_{AB} = \rho_1 \otimes \sigma_1 + \cdots + \rho_n \otimes \sigma_n,$$

where ρ_i's are matrices on the state space A and σ_i's are those on B. Suppose that we measure an observable of the physical system A

$$M = \sum_{k=1}^{m} kP_k.$$

The probability of getting the measurement outcome k is

$$\begin{aligned}
\mathrm{Tr}[\tau_{AB}(P_k \otimes I)] &= \mathrm{Tr}\left[\sum_{i=1}^{n}(\rho_i \otimes \sigma_i)(P_k \otimes I)\right] \\
&= \mathrm{Tr}\left[\sum_{i=1}^{n} \rho_i P_k \otimes \sigma_i\right] \\
&= \sum_{i=1}^{n} \mathrm{Tr}[\rho_i P_k \otimes \sigma_i] \\
&= \sum_{i=1}^{n} \mathrm{Tr}[\rho_i P_k]\mathrm{Tr}[\sigma_i] \\
&= \sum_{i=1}^{n} \mathrm{Tr}[\mathrm{Tr}[\sigma_i]\rho_i P_k] \\
&= \mathrm{Tr}\left[\left(\sum_{i=1}^{n} \mathrm{Tr}[\sigma_i]\rho_i\right) P_k\right]
\end{aligned}$$

The probability distribution of measurement outcomes (P_k) is the same as measuring the state

$$\sum_{i=1}^{n} \mathrm{Tr}[\sigma_i]\rho_i \tag{9}$$

of the system A. The state (9) is called the partial trace of τ_{AB} over B, and denoted by $\mathrm{Tr}_B[\tau_{AB}]$.

Exercise

1. Suppose that the system is in the state $|0\rangle$ with probability 0.5 and $|1\rangle$ with probability 0.5. Write the corresponding density matrix as a 2×2 matrix.
2. Suppose that the system is in the state $(|0\rangle + |1\rangle)/\sqrt{2}$ with probability 0.5 and $(|0\rangle - |1\rangle)/\sqrt{2}$ with probability 0.5. Write the corresponding density matrix as a 2×2 matrix.
3. Let P be an $n \times n$ projection matrix of rank 1 ($n \geq 2$). Show that $\mathrm{Tr}[P] = 1$. (Hint: A projection matrix of rank 1 can be written as $|\varphi\rangle\langle\varphi|$ with some vector $|\varphi\rangle$ with $\||\varphi\rangle\| = 1$.)
4. Let M be a 2×2 Hermitian matrix with its spectral decomposition $M = \lambda_1 P_1 + \lambda_2 P_2$ with $\lambda_1 \neq \lambda_2$. Show that $\mathrm{Tr} P_1 = \mathrm{Tr} P_2 = 1$ by using your answer to Problem 3. (Hint: What are the ranks of P_1 and P_2?)
5. Let $\rho = I/2$, where I is the 2×2 identity matrix. Suppose that the system is in state ρ and we measure the observable M given in Problem 4. Compute the probabilities of getting outcomes λ_1 and λ_2 by using your answer to Problem 4.

6. Let $|\Psi\rangle = (|0_A 1_B\rangle + |1_A 0_B\rangle)/\sqrt{2}$ be a state of systems A and B. Compute the partial trace of $|\Psi\rangle\langle\Psi|$ over B. $\{|0_A\rangle, |1_A\rangle\}$ and $\{|0_B\rangle, |1_B\rangle\}$ are orthonormal bases of A and B, respectively. The state $|\Psi\rangle$ corresponds to the situation in which the sender applied X to his qubit for the superdense coding.

7. Suppose that we measure the observable $M \otimes I$ of the system with state $|\Psi\rangle$. Compute the probabilities of getting outcomes λ_1 and λ_2 by using your answers of Problems 5 and 6. (Hint: What density matrix on the system A represents the state $|\Psi\rangle$?)

Hint to Q6

Let $|\Phi\rangle = (|0_A 0_B\rangle + |1_A 1_B\rangle)/\sqrt{2}$. We can compute the partial trace of $|\Phi\rangle\langle\Phi|$ over B as follows: The density matrix corresponding $|\Phi\rangle\langle\Phi|$ is

$$\begin{aligned} |\Phi\rangle\langle\Phi| \quad = \quad & \frac{1}{2}(|0_A\rangle\langle 0_A| \otimes |0_B\rangle\langle 0_B| + |1_A\rangle\langle 1_A| \otimes |1_B\rangle\langle 1_B| + \\ & |0_A\rangle\langle 1_A| \otimes |0_B\rangle\langle 1_B| + |1_A\rangle\langle 0_A| \otimes |1_B\rangle\langle 0_B|) \end{aligned} \tag{10}$$

Observe that $\mathrm{Tr}[|0_B\rangle\langle 1_B|] = \mathrm{Tr}[|1_B\rangle\langle 0_B|] = 0$ and $\mathrm{Tr}[|0_B\rangle\langle 0_B|] = \mathrm{Tr}[|1_B\rangle\langle 1_B|] = 1$.

Thus, the partial trace of (10) over B is

$$
\begin{aligned}
& \frac{1}{2}(\underbrace{\mathrm{Tr}[|0_B\rangle\langle 0_B|]}_{=1}|0_A\rangle\langle 0_A| + \underbrace{\mathrm{Tr}[|1_B\rangle\langle 1_B|]}_{=1}|1_A\rangle\langle 1_A| + \\
& \underbrace{\mathrm{Tr}[|0_B\rangle\langle 1_B|]}_{=0}|0_A\rangle\langle 1_A| + \underbrace{\mathrm{Tr}[|1_B\rangle\langle 0_B|]}_{=0}|1_A\rangle\langle 0_A|) \\
= \quad & \frac{1}{2}(|0_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|) \\
= \quad & I/2
\end{aligned}
$$