QIP Course 5: Quantum Superdense Coding

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Partial answers to the exercises

$$
\begin{aligned}
Z_1 &= |0\rangle\langle 0| - |1\rangle\langle 1|, \\
Z_1 \otimes I \otimes I &= |0\rangle\langle 0| \otimes I \otimes I - |1\rangle\langle 1| \otimes I \otimes I
\end{aligned}
$$

Define

$$
\begin{aligned}
|\Phi\rangle &= \frac{1}{2}[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + \\
&\quad |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)]
\end{aligned}
$$

Here we start computing the probability of getting outcome $+1$ by measuring Z_1 and the state after it.

$$
\begin{aligned}
(|0\rangle\langle 0| \otimes I \otimes I) 2|\Phi\rangle &= (|0\rangle\langle 0| \otimes I \otimes I) \\
&\quad [|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + \\
&\quad\ |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)]
\end{aligned}
$$

Since $|0\rangle\langle 0||0\rangle = |0\rangle$ and $|0\rangle\langle 0||1\rangle = 0$, we have

$$
\begin{aligned}
(|0\rangle\langle 0| \otimes I \otimes I)|00\rangle(\alpha|0\rangle + \beta|1\rangle) &= \text{unchanged}, \\
(|0\rangle\langle 0| \otimes I \otimes I)|01\rangle(\alpha|1\rangle + \beta|0\rangle) &= \text{unchanged}, \\
(|0\rangle\langle 0| \otimes I \otimes I)|10\rangle(\alpha|0\rangle - \beta|1\rangle) &= 0, \\
(|0\rangle\langle 0| \otimes I \otimes I)|11\rangle(\alpha|1\rangle - \beta|0\rangle) &= 0
\end{aligned}
$$

Thus,

$$(|0\rangle\langle 0| \otimes I \otimes I)|\Phi\rangle = \frac{1}{2}[\underbrace{|00\rangle(\alpha|0\rangle + \beta|1\rangle)}_{\text{length 1}} + \underbrace{|01\rangle(\alpha|1\rangle + \beta|0\rangle)}_{\text{length 1}}],$$

and its squared norm is 0.5, because the two terms in the last line are orthogonal to each other.
Therefore,

- the probability of getting $+1$ on measuring the observable Z_1 of the leftmost qubit is 0.5, and
- the state after measurement is

$$|\Phi_2\rangle = \frac{1}{\sqrt{2}}[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle)].$$

Here we start computing the conditional probability of getting outcome -1 by measuring Z_2 and the state after it.

$$I \otimes Z_2 \otimes I = I \otimes |0\rangle\langle 0| \otimes I - I \otimes |1\rangle\langle 1| \otimes I$$

$$\begin{aligned} & (I \otimes |0\rangle\langle 0| \otimes I)\sqrt{2}|\Phi_2\rangle \\ = \quad & (I \otimes |0\rangle\langle 0| \otimes I)[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle)] \end{aligned}$$

Since $|0\rangle\langle 0||0\rangle = |0\rangle$ and $|0\rangle\langle 0||1\rangle = 0$, we have

$$\begin{aligned}(I \otimes |0\rangle\langle 0| \otimes I)|00\rangle(\alpha|0\rangle + \beta|1\rangle) &= \text{unchanged},\\ (I \otimes |0\rangle\langle 0| \otimes I)|01\rangle(\alpha|1\rangle + \beta|0\rangle) &= 0\end{aligned}$$

Thus,

$$(I \otimes |0\rangle\langle 0| \otimes I)|\Phi_2\rangle = \frac{1}{\sqrt{2}}[|00\rangle(\alpha|0\rangle + \beta|1\rangle)],$$

and its squared norm is 0.5.
Therefore,

- the probability of getting $+1$ on measuring the observable Z_2 of the middle qubit is 0.5, and
- the state after measurement is

$$|00\rangle(\alpha|0\rangle + \beta|1\rangle).$$

Observe that the joint probability of the measurement outcome $(+1, +1)$ of (Z_1, Z_2) is $0.5 \times 0.5 = 0.25$.

Superdense coding

1 qubit can carry at most 1 bit of information.

$$\Uparrow$$

Since 2×2 matrix has at most 2 eigenvalues, the number of measurement outcomes of measuring 1 qubit is at most 2.

Superdense coding sends 2 bits of information by sending 1 qubit.

- The sender and receiver are spatially apart.
- They share

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Orthogonal states can be distinguished

$\{|\varphi_1\rangle, ..., |\varphi_n\rangle\}$: orthonormal basis of a linear space $\mathcal{H}$.
If we know the state of a system is in one of $\{|\varphi_1\rangle, ..., |\varphi_n\rangle\}$, then we can distinguish them, as follows:

$$A = \sum_{k=1}^{n} k|\varphi_k\rangle\langle\varphi_k| \tag{1}$$

- A is a Hermitian matrix.
- If the state is one of $\{|\varphi_1\rangle, ..., |\varphi_n\rangle\}$, then we can distinguish them by measuring A.

More formally, the state before measurement is $|\varphi_k\rangle$ **if** and **only if** the measurement outcome is k (Excercise).

Superdense coding 2

The sender applies either I, X, Z, or XZ to his physical system. This color represents the sender and this color represents the receiver.

$$(X \otimes I)\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|10\rangle + |01\rangle}{\sqrt{2}}, \tag{2}$$

$$(Z \otimes I)\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \tag{3}$$

$$(XZ \otimes I)\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|10\rangle - |01\rangle}{\sqrt{2}}. \tag{4}$$

The above three states and $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ form an orthonormal basis of the state space of 2 qubits (excercise).

Superdense coding 3

The sender sends his physical system to the receiver.
The receiver has 2 qubits, and the state of 2 qubits is either (2), (3), (4), or $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$.
Since they are orthogonal, they can be distinguished by measuring an appropriate observable.
The receiver can distinguish 4 states, and thus he/she can obtain two bits of information.

Exercise (60 min.?)

1. Show that the matrix (1) is Hermitian.
2. Derive the identity (4) in detail.
3. Show that the inner product of the vectors (3) and (2) is zero.
4. Prove that the measurement outcome is k **if** the state **before measurement** is $|\varphi_k\rangle$ in page 9.
5. Prove that the state **before measurement** is $|\varphi_k\rangle$ **if** the measurement outcome is k in page 9.

Your understanding only comes through mathematics and your hand computation, because the intuition of human being is useless and misleading in the quantum physics!

Warning: Computation will become 5 times harder in Shor's quantum factorization algorithm! The teleportation and the dense coding are the easiest in QIP. Please become familiar with the bra-ket notations and the tensor products, at this point. Please bring Matlab or Maple from Unit 8 (maybe tomorrow).