QIP Course 2: Basics of QIP

Ryutaroh Matsumoto

Nagoya University, Japan
Send your comments to ryutaroh.matsumoto@nagoya-u.jp

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The slides include figures from the http://openstaxcollege.org/ and the Wikipedia.

Contents of the slides

- Vector representation of quantum states
- Measurement

Minimal explanation of quantum mechanics

I will introduce the mathematical model of the quantum theory. It does not include Schrödinger's equation.

Schrödinger's equation is almost always explained in a course on quantum physics. I do not explain that. Schrödinger's equation is required when one wants to know the state of a quantum system as a function of time. We can understand the essential part of QIP without it.

State of a quantum system

Quantum system: whatever physical phenomenon. E.g. photon polarization.

The state of a quantum system is represented by a complex vector of length (norm) 1 in a complex linear space.

The dimension of the linear space associated with a quantum system is usually infinite dimensional.
Assumption: The dimension of linear space is always finite in this course.

Notation of vectors

$|\varphi\rangle$: column vector in the quantum physics
$\langle\varphi|$: the complex conjugate transpose of $|\varphi\rangle$.

Example of states of linear photon polarization

$|-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $||\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
$|/\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|\backslash\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
The direction of polarization is represented by that of state vector.

What is represented by a complex vector?

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix}$ represents the circular polarization, which means that polarization is rotating as the photon moves.

Measurement

Measurement of a quantum system = an action of extracting information from the system.
A Hermitian matrix represents how to measure a quantum system.

A complex square matrix M is *Hermitian* if $M = M^*$.

Eigenvalue and eigenspace

M: complex square matrix
A complex number λ is said to be an *eigenvalue* of M if there exists a nonzero vector $\vec{v}$ such that $M\vec{v} = \lambda\vec{v}$.

Eigenspace belonging to λ = $\{\vec{u} \mid M\vec{u} = \lambda\vec{u}\}$.

Projection onto a subspace

V: linear space
W: subspace of V
$W^\perp$: orthogonal complement of W in V

P_W: projection onto W
Any $\vec{v}$ can written uniquely as

$$\vec{v} = \vec{w}_1 + \vec{w}_2$$

with $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^\perp$.

$P_W(\vec{v}) = \vec{w}_1$.

How to compute the matrix representation of P_W

1. Find an orthonormal basis $\{|\psi_1\rangle, ..., |\psi_m\rangle\}$ of W.
2. $$P_W = |\psi_1\rangle\langle\psi_1| + \cdots + |\psi_m\rangle\langle\psi_m| \tag{1}$$

Spectral decomposition

M: Hermitian matrix
λ_i: i-th eigenvalue of M ($\lambda_i \neq \lambda_j$)
W_i: eigenspace belonging to λ_i
P_i: projection onto W_i.

$$M = \sum_i \lambda_i P_i$$

The above decomposition is called the spectral decomposition of M.

How to compute spectral decomposition

1. Compute all eigenvalues.
2. For each eigenspace W_i, find an orthonormal basis $\{|\psi_{i1}\rangle, ..., |\psi_{im}\rangle\}$ of W_i.
3. P_i is given by

$$P_i = \sum_{k=1}^{m} |\psi_{ik}\rangle\langle\psi_{ik}|.$$

Measurement

$\mathcal{H}$: linear space associated with a quantum system
Measurement is described by an observable A, which is a Hermitian matrix on $\mathcal{H}$.
Results of measuring the observable A = eigenvalues of A.

We cannot predict which measurement outcome is obtained before the measurement, e.g. the measurement of polarization. But we can calculate the probability of a measurement outcome.

Probability of getting a measurement outcome

The quantum system is in state $|\varphi\rangle$.
Measuring an observable A.
$\lambda_1, \ldots, \lambda_n$: eigenvalues of A.

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n.$$

The probability of getting λ_i as the measurement outcome =

$$\| P_i |\varphi\rangle \|^2 . \tag{2}$$

α: complex number with $|\alpha| = 1$
Since $|\varphi\rangle$ and $\alpha|\varphi\rangle$ give the same probability distribution of the measurement outcomes, they are physically indistinguishable. $|\varphi\rangle$ and $\alpha|\varphi\rangle$ represent the same quantum state.

Example of an observable

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $Z^* = Z$, it is Hermitian.

eigenvalue	eigenvector	projector
$+1$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
-1	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

The spectral decomposition of Z:

$$Z = +1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

If the state is $|\varphi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

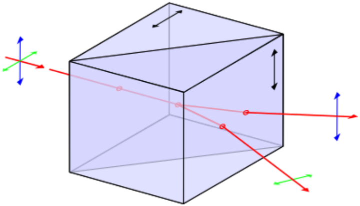
$$P_1|\varphi\rangle = |\varphi\rangle, P_2|\varphi\rangle = 0.$$

Probability of getting $+1$ as the measurement outcome is 1.
Probability of getting -1 as the measurement outcome is 0.

If polarization is represented as in p. 6, Z represents the measurement by the slit | (or –).

Nondestructive measurement

When we measure polarization of a photon by a slit, the photon can be absorbed.
There is measurement with which the measured quantum system does not disappear. Such measurement is called *nondestructive measurement*.
Nondestructive measurement of photon polarization can be done by the prism consisting of calcite ($CaCO_3$) crystal like



Excerpted from
`http://commons.wikimedia.org/wiki/File:Wollaston-prism.svg`.

State after nondestructive measurement

Quantum state is changed by nondestructive measurement.
Measuring an observable A of a system with state $|\varphi\rangle$ nondestructively

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n.$$

After getting a measurement outcome λ_i, the state become

$$\frac{P_i|\varphi\rangle}{||P_i|\varphi\rangle||}.$$

Example of nondestructive measurement

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, |\varphi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, |a|^2 + |b|^2 = 1.$$

$$\frac{P_1|\varphi\rangle}{||P_1|\varphi\rangle||} = \begin{pmatrix} a/|a| \\ 0 \end{pmatrix}$$

is equivalent to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which represents the – polarization

$$\frac{P_2|\varphi\rangle}{||P_2|\varphi\rangle||} = \begin{pmatrix} 0 \\ b/|b| \end{pmatrix}$$

is equivalent to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which represents the | polarization.

Above equations says that after measuring whether the polarization is – or |, polarization becomes - or | according to the measurement outcome.

Exercises (60 min.??)

Please discuss them with other students. You are also welcomed to talk with the lecturer.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, |\varphi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

1. Is X a Hermitian matrix?
2. Compute the spectral decomposition of X.
3. Suppose that one measures the observable X of the system with state $|\varphi\rangle$. For each measurement outcome, compute its probability and the state after getting the outcome.
4. If photon polarization is represented as page 6, which polarizations are measured by X?

5. (Optional for non-math students) Prove

$$\sum_{j=1}^{n} \left\| P_j |\varphi\rangle \right\|^2 = 1,$$

where P_i and $|\varphi\rangle$ are as defined in Eq. (2). You must not assume that $\left\| P_j |\varphi\rangle \right\|^2$ forms a probability distribution, which you are requested to verify. You must not assume that P_j can be written as $|\varphi\rangle\langle\varphi|$ for some vector φ, because an eigenvalue can have two or more linearly independent eigenvectors.

6. (Optional for non-math students). Prove that Eq. (1) is the projection onto W in the sense of page 9.