Now, since $f(\boldsymbol{x})$ is convex, $f(\boldsymbol{x}_k) \geq f(\boldsymbol{y}_k) + \langle \boldsymbol{\nabla f}(\boldsymbol{y}_k), \boldsymbol{x}_k - \boldsymbol{y}_k \rangle$, and multiplying this inequality by $(1-\alpha_k)$ we have:

$$\phi_{k+1}^* \geq f(\boldsymbol{y}_k) - \frac{\alpha_k^2}{2\gamma_{k+1}} \|\boldsymbol{\nabla f}(\boldsymbol{y}_k)\|_2^2 + (1-\alpha_k)\langle \boldsymbol{\nabla f}(\boldsymbol{y}_k), \frac{\alpha_k\gamma_k}{\gamma_{k+1}}(\boldsymbol{v}_k - \boldsymbol{y}_k) + \boldsymbol{x}_k - \boldsymbol{y}_k \rangle + \frac{\alpha_k(1-\alpha_k)\gamma_k\mu}{2\gamma_{k+1}}\|\boldsymbol{y}_k - \boldsymbol{v}_k\|_2^2.$$

Recall that since $\boldsymbol{\nabla f}$ is L-Lipschitz continuous, if we apply Lemma 3.6 to $\boldsymbol{y}_k$ and $\boldsymbol{x}_{k+1} = \boldsymbol{y}_k - \frac{1}{L}\boldsymbol{\nabla f}(\boldsymbol{y}_k)$, we obtain

$$f(\boldsymbol{y}_k) - \frac{1}{2L}\|\boldsymbol{\nabla f}(\boldsymbol{y}_k)\|_2^2 \geq f(\boldsymbol{x}_{k+1}).$$

Therefore, if we impose

$$\frac{\alpha_k\gamma_k}{\gamma_{k+1}}(\boldsymbol{v}_k - \boldsymbol{y}_k) + \boldsymbol{x}_k - \boldsymbol{y}_k = \boldsymbol{0}$$

it justifies our choice for $\boldsymbol{y}_k$. And putting

$$\frac{\alpha_k^2}{2\gamma_{k+1}} = \frac{1}{2L}$$

it justifies our choice for α_k. Since $\frac{\alpha_k(1-\alpha_k)\gamma_k\mu}{\gamma_{k+1}} \geq 0$, we finally obtain $\phi_{k+1}^* \geq f(\boldsymbol{x}_{k+1})$ as wished. ∎

The above theorem suggests an algorithm to minimize $f \in \mathcal{S}_{\mu,L}^{1,1}(\mathbb{R}^n)$.

Notice that in the following method, we don't need the estimated sequence anymore.

Generic Scheme for the Nesterov's Optimal Gradient Method

```
Step 0: Choose $\boldsymbol{x}_0 \in \mathbb{R}^n$, let $\gamma_0 > 0$ such that $L \geq \gamma_0 \geq \mu \geq 0$.
        Set $\boldsymbol{v}_0 := \boldsymbol{x}_0$ and $k := 0$.
Step 1: Compute $\alpha_k \in (0,1]$ from the equation $L\alpha_k^2 = (1-\alpha_k)\gamma_k + \alpha_k\mu$.
Step 2: Set $\gamma_{k+1} := (1-\alpha_k)\gamma_k + \alpha_k\mu$, $\boldsymbol{y}_k := \frac{\alpha_k\gamma_k\boldsymbol{v}_k + \gamma_{k+1}\boldsymbol{x}_k}{\gamma_k + \alpha_k\mu}$.
Step 3: Compute $f(\boldsymbol{y}_k)$ and $\boldsymbol{\nabla f}(\boldsymbol{y}_k)$.
Step 4: Find $\boldsymbol{x}_{k+1}$ such that $f(\boldsymbol{x}_{k+1}) \leq f(\boldsymbol{y}_k) - \frac{1}{2L}\|\boldsymbol{\nabla f}(\boldsymbol{y}_k)\|_2^2$ using "line search".
Step 5: Set $\boldsymbol{v}_{k+1} := \frac{(1-\alpha_k)\gamma_k\boldsymbol{v}_k + \alpha_k\mu\boldsymbol{y}_k - \alpha_k\boldsymbol{\nabla f}(\boldsymbol{y}_k)}{\gamma_{k+1}}$, $k := k+1$ and go to Step 1.
```

Theorem 8.6 Consider $f \in \mathcal{S}_{\mu,L}^{1,1}(\mathbb{R}^n)$, possible with $\mu = 0$ (which means that $f \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$). The generic scheme of the Nesterov's optimal gradient method generates a sequence $\{\boldsymbol{x}_k\}_{k=0}^\infty$ such that

$$\begin{aligned} f(\boldsymbol{x}_k) - f(\boldsymbol{x}^*) &\leq \lambda_k \left[f(\boldsymbol{x}_0) + \frac{\gamma_0}{2}\|\boldsymbol{x}^* - \boldsymbol{x}_0\|_2^2 - f(\boldsymbol{x}^*) \right] \\ &\leq \min\left\{ \left(1 - \sqrt{\frac{\mu}{L}}\right)^k, \frac{4L}{(2\sqrt{L} + k\sqrt{\gamma_0})^2} \right\} \left[f(\boldsymbol{x}_0) + \frac{\gamma_0}{2}\|\boldsymbol{x}^* - \boldsymbol{x}_0\|_2^2 - f(\boldsymbol{x}^*) \right], \end{aligned}$$

where $\alpha_{-1} = 0$ and $\lambda_k = \prod_{i=-1}^{k-1}(1-\alpha_i)$.

In other words, the sequence $\{f(\boldsymbol{x}_k) - f(\boldsymbol{x}^*)\}_{k=0}^\infty$ converges R-sublinearly to zero if $\mu = 0$ and R-linearly to zero if $\mu > 0$.

In addition, if $\mu > 0$,

$$\begin{aligned} \|\boldsymbol{x}_k - \boldsymbol{x}^*\|^2 &\leq \frac{2}{\mu}\lambda_k \left[f(\boldsymbol{x}_0) + \frac{\gamma_0}{2}\|\boldsymbol{x}^* - \boldsymbol{x}_0\|_2^2 - f(\boldsymbol{x}^*) \right] \\ &\leq \frac{2}{\mu}\min\left\{ \left(1 - \sqrt{\frac{\mu}{L}}\right)^k, \frac{4L}{(2\sqrt{L} + k\sqrt{\gamma_0})^2} \right\} \left[f(\boldsymbol{x}_0) + \frac{\gamma_0}{2}\|\boldsymbol{x}^* - \boldsymbol{x}_0\|_2^2 - f(\boldsymbol{x}^*) \right]. \end{aligned}$$

That is, $\{\|\boldsymbol{x}_k - \boldsymbol{x}^*\|^2\}_{k=0}^\infty$ converges R-linearly to zero.

Proof:

The first inequality is obvious from the definitions and Lemma 8.2.

We already know that $\alpha_k \geq \sqrt{\frac{\mu}{L}}$ $(k = 0, 1, \ldots)$ (see proof of Theorem 8.5), therefore,

$$\lambda_k = \prod_{i=-1}^{k-1}(1-\alpha_i) = \prod_{i=0}^{k-1}(1-\alpha_i) \leq \left(1 - \sqrt{\frac{\mu}{L}}\right)^k,$$

which only has an effect if $\mu > 0$. For the case $\mu = 0$, we already proved in Theorem 8.5.

For $\mu > 0$, using the definition of strong convexity of $f(\boldsymbol{x})$, we obtain the upper bound for $\|\boldsymbol{x}_k - \boldsymbol{x}^*\|_2^2$. ▮

Corollary 8.7 Consider $f \in \mathcal{S}^{1,1}_{\mu,L}(\mathbb{R}^n)$, possible with $\mu = 0$ (which means that $f \in \mathcal{F}^{1,1}_L(\mathbb{R}^n)$). If we take $\gamma_0 = L$, the generic scheme of the Nesterov's optimal gradient method generates a sequence $\{\boldsymbol{x}_k\}_{k=0}^{\infty}$ such that

$$f(\boldsymbol{x}_k) - f(\boldsymbol{x}^*) \leq L \min\left\{\left(1 - \sqrt{\frac{\mu}{L}}\right)^k, \frac{4}{(k+2)^2}\right\} \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2^2.$$

In other words, the sequence $\{f(\boldsymbol{x}_k) - f(\boldsymbol{x}^*)\}_{k=0}^{\infty}$ converges R-sublinearly to zero if $\mu = 0$ and R-linearly to zero if $\mu > 0$.

In the particular case of $\mu > 0$, we have the following inequality:

$$\|\boldsymbol{x}_k - \boldsymbol{x}^*\|_2^2 \leq \frac{2L}{\mu} \min\left\{\left(1 - \sqrt{\frac{\mu}{L}}\right)^k, \frac{4}{(k+2)^2}\right\} \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2^2.$$

That means that the sequence $\{\|\boldsymbol{x}_k - \boldsymbol{x}^*\|_2^2\}_{k=0}^{\infty}$ converges R-linearly to zero.

Proof:

The two inequalities follow from the previous theorem, $f(\boldsymbol{x}_0) - f(\boldsymbol{x}^*) \leq \langle \boldsymbol{\nabla f}(\boldsymbol{x}^*), \boldsymbol{x}_0 - \boldsymbol{x}^* \rangle + \frac{L}{2}\|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2^2$, and the fact that $\boldsymbol{\nabla f}(\boldsymbol{x}^*) = \boldsymbol{0}$. ▮

Now, instead of doing a line search at Step 4 of the generic scheme for the Nesterov's optimal gradient method, let us consider the constant step size iteration $\boldsymbol{x}_{k+1} := \boldsymbol{y}_k - \frac{1}{L}\boldsymbol{\nabla f}(\boldsymbol{y}_k)$ (see proof of Theorem 8.5). From the calculations given at Exercise 1, we arrive to the following simplified scheme. Hereafter, we assume that $L > \mu$ to exclude the trivial case $L = \mu$ with finished in one iteration.

Constant Step Scheme for the Nesterov's Optimal Gradient Method

```
Step 0: Choose $\boldsymbol{x}_0 \in \mathbb{R}^n$, $\alpha_0 \in (0,1)$ such that $\frac{\alpha_0(\alpha_0 L - \mu)}{1-\alpha_0} > 0$, $\mu \leq \frac{\alpha_0(\alpha_0 L - \mu)}{1-\alpha_0} \leq L$,
        set $\boldsymbol{y}_0 := \boldsymbol{x}_0$ and $k := 0$.
Step 1: Compute $\boldsymbol{\nabla f}(\boldsymbol{y}_k)$.
Step 2: Set $\boldsymbol{x}_{k+1} := \boldsymbol{y}_k - \frac{1}{L}\boldsymbol{\nabla f}(\boldsymbol{y}_k)$.
Step 3: Compute $\alpha_{k+1} \in (0,1)$ from the equation $\alpha_{k+1}^2 = (1-\alpha_{k+1})\alpha_k^2 + \frac{\mu}{L}\alpha_{k+1}$.
Step 4: Set $\beta_k := \frac{\alpha_k(1-\alpha_k)}{\alpha_k^2 + \alpha_{k+1}}$.
Step 5: Set $\boldsymbol{y}_{k+1} := \boldsymbol{x}_{k+1} + \beta_k(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k)$, $k := k+1$ and go to Step 1.
```

Observe that the sequences $\{\boldsymbol{x}_k\}_{k=0}^{\infty}$ and $\{\boldsymbol{y}_k\}_{k=0}^{\infty}$ generated by the "Generic Scheme" and the "Constant Step Scheme" are exactly the same[4] if we choose $\boldsymbol{x}_{k+1} := \boldsymbol{y}_k - \frac{1}{L}\boldsymbol{\nabla f}(\boldsymbol{y}_k)$ in the former method. Therefore, the result of Theorem 8.6 is still valid for $\gamma_0 := \alpha_0(\alpha_0 L - \mu)/(1-\alpha_0)$.

[4] strictly speaking, there is a one index difference between $\boldsymbol{y}_k$'s on these two methods due to the order $\boldsymbol{y}_k$ is defined in the loop.

Also, if we further impose $\gamma_0 = \alpha_0(\alpha_0 L - \mu)/(1-\alpha_0) = L$, we will have the rate of convergence of Theorem 8.7.

Theorem 8.8 Consider $f \in \mathcal{S}^{1,1}_{\mu,L}(\mathbb{R}^n)$, possible with $\mu = 0$ (which means that $f \in \mathcal{F}^{1,1}_L(\mathbb{R}^n)$). The constant step scheme of the Nesterov's optimal gradient method generates a sequence $\{\boldsymbol{x}_k\}_{k=0}^{\infty}$ such that

$$f(\boldsymbol{x}_k) - f(\boldsymbol{x}^*) \leq L \min\left\{ \left(1 - \sqrt{\frac{\mu}{L}}\right)^k, \frac{4}{(k+2)^2} \right\} \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2^2,$$

and

$$\|\boldsymbol{x}_k - \boldsymbol{x}^*\|_2^2 \leq \frac{2L}{\mu} \min\left\{ \left(1 - \sqrt{\frac{\mu}{L}}\right)^k, \frac{4}{(k+2)^2} \right\} \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2^2.$$

This means that the method is "optimal" for the class of functions $\mathcal{F}^{1,1}_L(\mathbb{R}^n)$, and $\mathcal{S}^{1,1}_{\mu,L}(\mathbb{R}^n)$.

Proof: Since the inequalities above are already shown in the previous Corollary 8.7, it remains to show the "optimality" of the methods for each class of functions.

For the case $\mu = 0$, the "optimality" of the method is obvious from Theorem 6.1.

Let us analyze the case when $\mu > 0$. From Theorem 6.2, we know that we can find a function $f \in \mathcal{S}^{\infty,1}_{\mu,L}(\ell^2)$ such that

$$f(\boldsymbol{x}_k) - f(\boldsymbol{x}^*) \geq \frac{\mu}{2} \left(\frac{\sqrt{L/\mu} - 1}{\sqrt{L/\mu} + 1} \right)^{2k} \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2^2 \geq \frac{\mu}{2} \exp\left(-\frac{4k}{\sqrt{L/\mu} - 1} \right) \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2^2,$$

where the second inequality follows from $\ln(\frac{a-1}{a+1}) = -\ln(\frac{a+1}{a-1}) \geq 1 - \frac{a+1}{a-1} = -\frac{2}{a-1}$, for $a \in (1, +\infty)$. Therefore, the worst case bound to find $\boldsymbol{x}_k$ such that $f(\boldsymbol{x}_k) - f(\boldsymbol{x}^*) < \varepsilon$ can not be better than

$$k > \frac{\sqrt{L/\mu} - 1}{4} \left(\ln \frac{1}{\varepsilon} + \ln \frac{\mu}{2} + 2 \ln \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2 \right).$$

On the other hand, from the inequality above

$$f(\boldsymbol{x}_k) - f(\boldsymbol{x}^*) \leq L \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2^2 \left(1 - \sqrt{\frac{\mu}{L}}\right)^k \leq L \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2^2 \exp\left(-\frac{k}{\sqrt{L/\mu}} \right),$$

where the second inequality follows from $\ln(1-a) \leq -a$ for $a < 1$. Therefore, we can guarantee $f(\boldsymbol{x}_k) - f(\boldsymbol{x}^*) < \varepsilon$ for $k > \sqrt{L/\mu}\left(\ln \frac{1}{\varepsilon} + \ln L + 2 \ln \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2\right)$.

Now, let us analize the sequences $\{\boldsymbol{x}_k\}_{k=0}^{\infty}$ generated by the method. Again from Theorem 6.2, we can find a function $f \in \mathcal{S}^{\infty,1}_{\mu,L}(\ell^2)$ such that

$$\|\boldsymbol{x} - \boldsymbol{x}^*\|_2^2 \geq \left(\frac{\sqrt{L/\mu} - 1}{\sqrt{L/\mu} + 1} \right)^{2k} \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2^2 \geq \exp\left(-\frac{4k}{\sqrt{L/\mu} - 1} \right) \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2^2.$$

Therefore, the worst case bound to find $\boldsymbol{x}_k$ such that $\|\boldsymbol{x}_k - \boldsymbol{x}^*\|_2^2 < \varepsilon$ can not be better than

$$k > \frac{\sqrt{L/\mu} - 1}{4} \left(\ln \frac{1}{\varepsilon} + 2 \ln \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2 \right).$$

On the other hand, from the inequality above

$$\|\boldsymbol{x}_k - \boldsymbol{x}^*\|_2^2 \leq \frac{2L}{\mu} \left(1 - \sqrt{\frac{\mu}{L}}\right)^k \leq \frac{2L}{\mu} \exp\left(-\frac{k}{\sqrt{L/\mu}} \right).$$

Therefore, we can guarantee $\|\boldsymbol{x}_k - \boldsymbol{x}^*\|_2^2 < \varepsilon$ for $k > \sqrt{L/\mu}\left(\ln \frac{1}{\varepsilon} + \ln 2L - \ln \mu + 2 \ln \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2\right)$.

This shows that the constant step scheme for the Nesterov's gradient method is an optimal method in terms of complexity for the dominant term $\ln(\varepsilon^{-1})$. ∎

Remark 8.9 Many times, you will find in articles that a method has "optimal rate of convergence". In our case, if we apply the constant step scheme for the Nesterov's optimal gradient method to $\min_{\boldsymbol{x}\in\mathbb{R}^n} f(\boldsymbol{x})$, the number of iterations of this method to obtain $f(\boldsymbol{x}_k) - f(\boldsymbol{x}^*) < \varepsilon$ is $k = k(L, \boldsymbol{x}_0, \boldsymbol{x}^*, \varepsilon) = \mathcal{O}\left(\sqrt{\frac{L\|\boldsymbol{x}_0-\boldsymbol{x}^*\|_2^2}{\varepsilon}}\right)$ and $k = k(L, \mu, \boldsymbol{x}_0, \boldsymbol{x}^*, \varepsilon) = \mathcal{O}\left(\sqrt{\frac{L}{\mu}} \ln \frac{L\|\boldsymbol{x}_0-\boldsymbol{x}^*\|_2^2}{\varepsilon}\right)$ for $f(\boldsymbol{x}) \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$ and $\mathcal{S}_{L,\mu}^{1,1}(\mathbb{R}^n)$, respectively.

It is extremely important to note that this value is the maximum number of iterations in the worse case scenario.

To obtain the total complexity of the method, you need to multiply the above number by the number of floating-point operations per iteration. This value also vary according to the method.

8.1 Discussion on Particular Cases

8.1.1 Nesterov's Optimal Gradient Method for Smooth (Differentiable) Strongly Convex Functions

In this case, we have $\mu > 0$ and choosing $\gamma_0 := \alpha_0(\alpha_0 L - \mu)/(1-\alpha_0) = \mu$, we can have further simplifications:

$$\alpha_k = \sqrt{\frac{\mu}{L}}, \quad \beta_k = \frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}}.$$

```
Nesterov's Optimal Gradient Method for Smooth Strongly Convex Function
Step 0:  Choose $\boldsymbol{x}_0 \in \mathbb{R}^n$, set $\boldsymbol{y}_0 := \boldsymbol{x}_0$ and $k := 0$.
Step 1:  Compute $\boldsymbol{\nabla f}(\boldsymbol{y}_k)$.
Step 2:  Set $\boldsymbol{x}_{k+1} := \boldsymbol{y}_k - \frac{1}{L}\boldsymbol{\nabla f}(\boldsymbol{y}_k)$.
Step 3:  Set $\boldsymbol{y}_{k+1} := \boldsymbol{x}_{k+1} + \frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k)$, $k := k+1$ and go to Step 1.
```

8.1.2 Optimal Gradient Method for Smooth (Differentiable) Convex Functions

In the case $\mu = 0$, there are much simpler variation of the method[5].

```
Nesterov's Original Optimal Gradient Method for Smooth Convex Function
Step 0:  Choose $\boldsymbol{x}_0 \in \mathbb{R}^n$, set $\boldsymbol{y}_0 := \boldsymbol{x}_0$, $t_0 := 1$, and $k := 0$.
Step 1:  Compute $\boldsymbol{\nabla f}(\boldsymbol{y}_k)$.
Step 2:  Set $\boldsymbol{x}_{k+1} := \boldsymbol{y}_k - \frac{1}{L}\boldsymbol{\nabla f}(\boldsymbol{y}_k)$.
Step 3:  $t_{k+1} := \dfrac{1 + \sqrt{1 + 4t_k^2}}{2}$.
Step 4:  Set $\boldsymbol{y}_{k+1} := \boldsymbol{x}_{k+1} + \dfrac{t_k - 1}{t_{k+1}}(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k)$, $k := k+1$ and go to Step 1.
```

Moreover, there is a simpler variant of this method.

```
Variant of Nesterov's Optimal Gradient Method for Smooth Convex Function
Step 0:  Choose $\boldsymbol{x}_0 \in \mathbb{R}^n$, set $\boldsymbol{y}_0 := \boldsymbol{x}_0$ and $k := 1$.
Step 1:  Compute $\boldsymbol{\nabla f}(\boldsymbol{y}_{k-1})$.
Step 2:  Set $\boldsymbol{x}_k := \boldsymbol{y}_{k-1} - \frac{1}{L}\boldsymbol{\nabla f}(\boldsymbol{y}_{k-1})$.
Step 3:  Set $\boldsymbol{y}_k := \boldsymbol{x}_k + \dfrac{k-1}{k+2}(\boldsymbol{x}_k - \boldsymbol{x}_{k-1})$, $k := k+1$ and go to Step 1.
```

[5] Y. Nesterov, "A method for solving the convex programming problem with convergence rate $\mathcal{O}(1/k^2)$," *Dokl. Akad. Nauk SSSR* **269** (1983), pp. 543–547. It also has a scheme to estimate L in the case this constant in unknown.

All of above methods generate sequence $\{\boldsymbol{x}_k\}_{k=0}^{\infty}$ such that

$$f(\boldsymbol{x}_k) - f(\boldsymbol{x}^*) \leq \frac{4L\|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2^2}{(k+1)^2}.$$

for $f \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$.

Recently, it was shown that an extension of this method guarantee a $o(k^{-2})$ convergence for $f(\boldsymbol{x}_k) - f(\boldsymbol{x}^*)$ by Attouch and Peypouquet[6].

Kim-Fessler's Optimal Gradient Method for Smooth Convex Function

```
Step 0:  Choose $\boldsymbol{x}_0 \in \mathbb{R}^n$, set $\boldsymbol{y}_0 := \boldsymbol{x}_0$, $t_0 := 1$, and $k := 0$.
Step 1:  Compute $\boldsymbol{\nabla f}(\boldsymbol{y}_k)$.
Step 2:  Set $\boldsymbol{x}_{k+1} := \boldsymbol{y}_k - \frac{1}{L}\boldsymbol{\nabla f}(\boldsymbol{y}_k)$.
Step 3:  $t_{k+1} := \begin{cases} \frac{1+\sqrt{1+4t_k^2}}{2}, & \text{if } k < N-2 \\ \frac{1+\sqrt{1+8t_k^2}}{2}, & \text{if } k = N-1 \end{cases}$.
Step 4:  Set $\boldsymbol{y}_{k+1} := \boldsymbol{x}_{k+1} + \frac{t_k - 1}{t_{k+1}}(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k) + \frac{t_k}{t_{k+1}}(\boldsymbol{x}_{k+1} - \boldsymbol{y}_k)$, $k := k+1$ and go to Step 1.
```

It can be shown that the Kim-Fessler's method generate sequence $\{\boldsymbol{x}_k\}_{k=0}^{N}$ such that

$$f(\boldsymbol{x}_N) - f(\boldsymbol{x}^*) \leq \frac{2L\|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2^2}{(N+2)^2}.$$

for $f \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$[7].

8.2 Exercises

1. We want to justify the Constant Step Scheme of the Optimal Gradient Method. This is a particular case of the General Scheme for the Optimal Gradient Method for the following choice:

$$\begin{aligned} \gamma_{k+1} &:= L\alpha_k^2 = (1-\alpha_k)\gamma_k + \alpha_k\mu \\ \boldsymbol{y}_k &= \frac{\alpha_k\gamma_k\boldsymbol{v}_k + \gamma_{k+1}\boldsymbol{x}_k}{\gamma_k + \alpha_k\mu} \\ \boldsymbol{x}_{k+1} &= \boldsymbol{y}_k - \frac{1}{L}\boldsymbol{\nabla f}(\boldsymbol{y}_k) \\ \boldsymbol{v}_{k+1} &= \frac{(1-\alpha_k)\gamma_k\boldsymbol{v}_k + \alpha_k\mu\boldsymbol{y}_k - \alpha_k\boldsymbol{\nabla f}(\boldsymbol{y}_k)}{\gamma_{k+1}}. \end{aligned}$$

   (a) Show that $\boldsymbol{v}_{k+1} = \boldsymbol{x}_k + \frac{1}{\alpha_k}(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k)$.

   (b) Show that $\boldsymbol{y}_{k+1} = \boldsymbol{x}_{k+1} + \beta_k(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k)$ for $\beta_k = \frac{\alpha_{k+1}\gamma_{k+1}(1-\alpha_k)}{\alpha_k(\gamma_{k+1}+\alpha_{k+1}\mu)}$.

   (c) Show that $\beta_k = \frac{\alpha_k(1-\alpha_k)}{\alpha_k^2+\alpha_{k+1}}$.

   (d) Explain why $\alpha_{k+1}^2 = (1-\alpha_{k+1})\alpha_k^2 + \frac{\mu}{L}\alpha_{k+1}$.

[6] Hedy Attouch and Juan Peypouquet, "The rate of convergence of Nesterovs accelerated forward-backward method is actually faster than $1/k^2$," *SIAM Journal on Optimization* **26** (2016), pp. 1824-1834.

[7] Donghwan Kim and Jeffrey A. Fessler, "Optimized first-order methods for smooth convex minimization," *Mathematical Programming* **159** (2016), pp. 81–107.