Graph Matching

prepared and Instructed by

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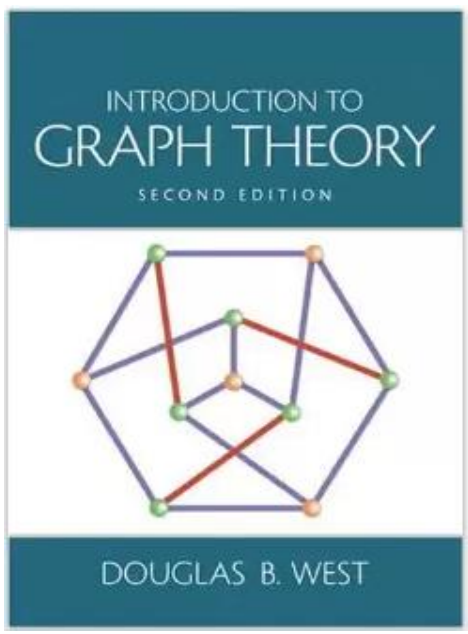
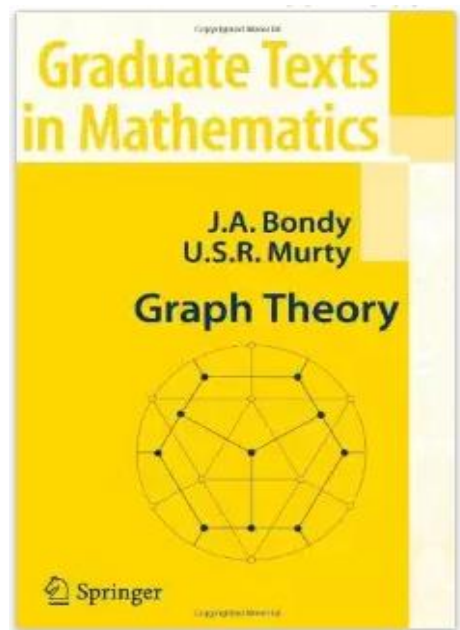
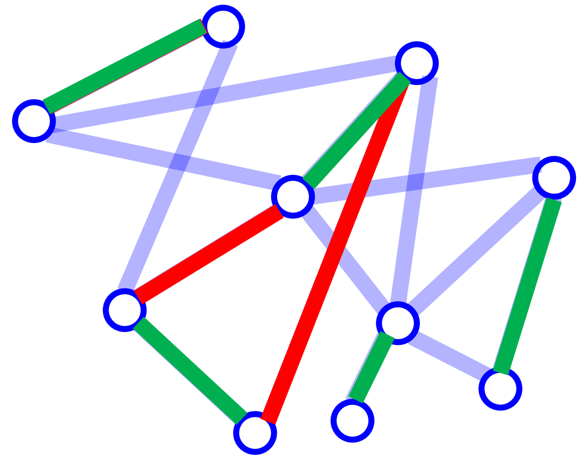




Matching in Bipartite Graphs

A **matching** in an undirected graph G is a set of pairwise disjoint edges.

A **perfect matching** consumes (saturates) all G's vertices. Also called **complete matching**.



$K_{n,n}$ has $n!$ perfect matchings. (why?)

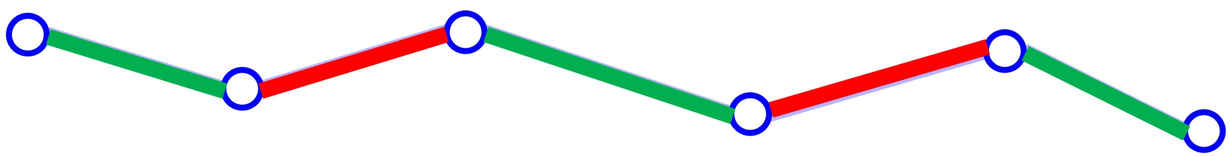
K_{2n+1} has no perfect matchings. (why?)

K_{2n} has $(2n)!/(2^n n!)$ perfect matchings. (why?)

Maximum Matching

A **maximal** matching is obtained by iteratively enlarging the matching with a disjoint edge until saturation.

A **maximum** matching is a matching of largest size. It is necessarily maximal.

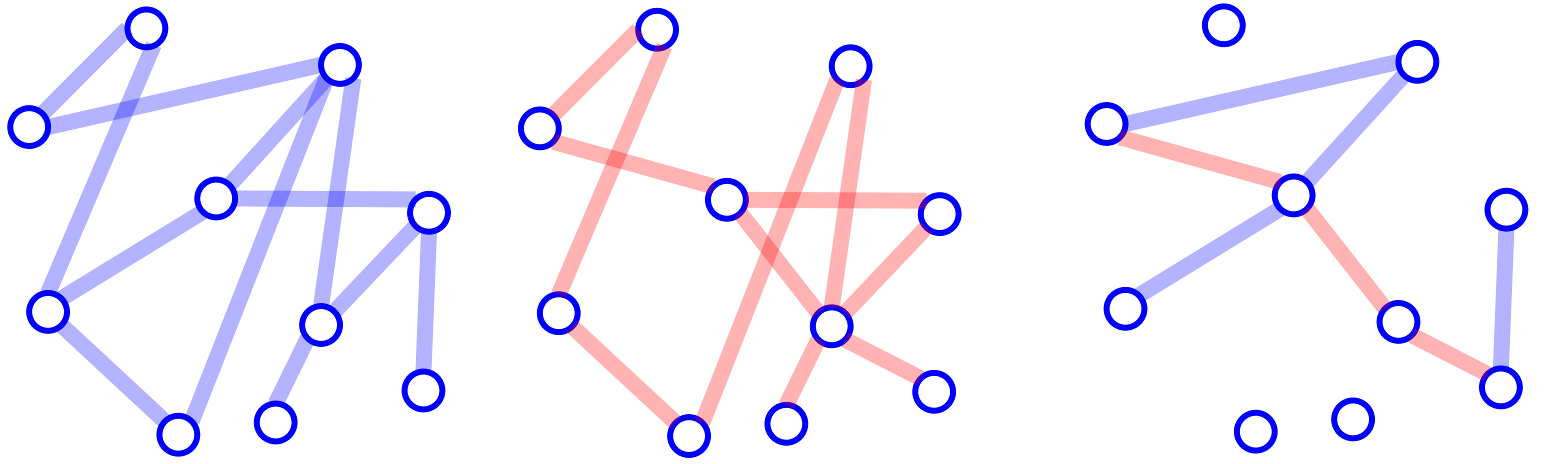


Given a matching M, an ***M-alternating path*** P is alternating between edges in M and edges not in M.

Let P's end vertices be not in M. Replacement of M's edges with $E(P) - M$ produces a matching M' such that $|M'| = |M| + 1$, called ***M*-augmentation**.

Maximum matching has no augmentation path. (why?)

Symmetric Difference:



$G(V, E_G)$ $\quad$ $H(V, E_H)$ $\quad$ $G \Delta H \triangleq F(V, E_G \oplus E_H)$

Symmetric difference is used also for matching.

If M and M' are two matchings then $M \Delta M' = (M \cup M') - (M \cap M')$.

Theorem. (Berge 1957) A matching M in a graph G is a maximum matching iff G has no M-augmentation path.

Proof. Maximum => no M-augmentation. As if G would have M-augmentation path, M could not be maximum.

For no M-augmentation => maximum, suppose that M is not maximum. We construct an M-augmentation.

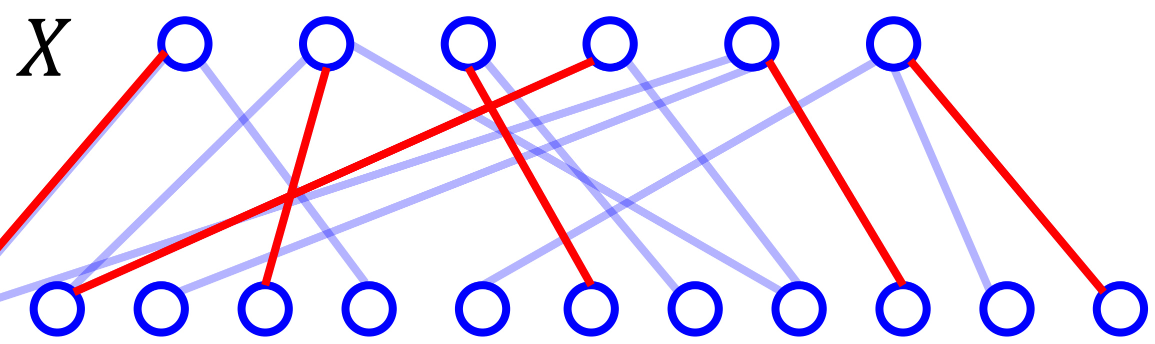
Consider a matching M', $|M'| > |M|$, and Let F be the spanning subgraph of G with $E(F) = M \Delta M'$.

M and M' are matchings so a vertex of F has degree 2 at most. F has therefore only disjoint paths and cycles, and cycles must be of even lengths. (why?)

Since $|M'| > |M|$ there is an edge alternating path with more edges of M' than M, and consequently there is an M-augmentation in G. ■

Hall's Matching Conditions

Y applicants apply for X jobs, $|Y| \gg |X|$. Each applicant applies for a few jobs. Can all the jobs be assigned?

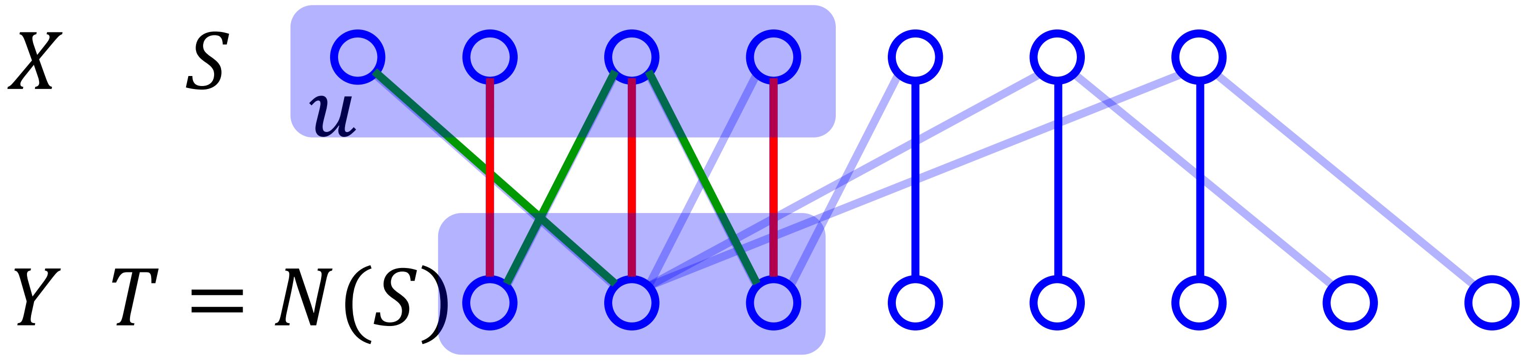


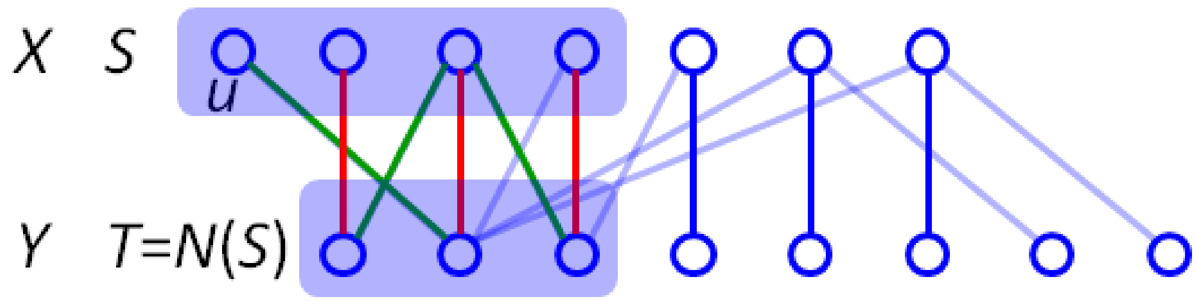
Denote $N(S)$ the neighbors in Y of $S \subset X$. $|N(S)| \geq |S|$ is clearly necessary for a matching saturating X.

Theorem. (Hall 1935) If $G[X, Y]$ is bipartite then G has a bipartition matching **saturating** X iff $|N(S)| \geq |S|$ for all $S \subseteq X$.

Proof. Sufficiency. Let M be **maximum** and for each $S \subseteq X$, there is $|N(S)| \geq |S|$. Let X be not saturated. There exists therefore $u \in X$, not M-saturated.

We will find S contradicting the theorem's hypothesis.

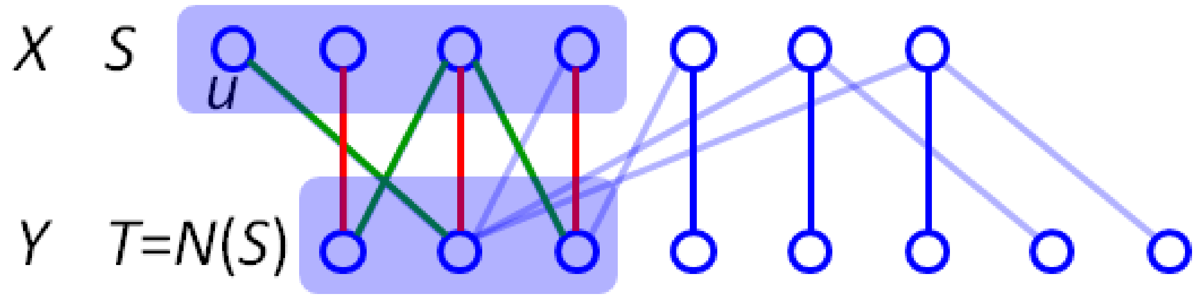




Let $S \subset X$ and $T \subset Y$ be reachable from u by ***M*-alternating** paths. We claim that M matches T with $S - u$.

The paths reach Y by edges not in M and X by M's edges. Since M is maximum, there is no M-augmentation paths, so every vertex of T is M-saturated.

Every $y \in T$ extends via M to a vertex in S. Also, $S - u$ is reached by M from T, thus $|T| = |S - u| = |S| - 1$.



The matching between T and $S-u$ implies $T \subseteq N(S)$.

In fact, $T = N(S)$. If there was an edge from S to a vertex $y \in Y - T$, it could not be in M, yielding an alternating path to y, contradicting $y \notin T$.

Therefore, $|N(S)| = |T| = |S| - 1 < |S|$, which is a contradiction of the theorem's hypothesis. ■

For $|X| = |Y|$, Hall's Theorem is the **Marriage Theorem**, proved originally by Frobenius in 1917, for a set of n men and n women.

If also every man is **compatible** (mutual preference) with k women and vice versa, there exists a perfect matching of **compatible** pairs (perfect matrimonial ☺).

Corollary. Every k-regular bipartite graph ($k > 0$) has a perfect matching.

Proof. Let X,Y be the bipartition. Counting **edges** from X to Y and from Y to X yields $k|X| = k|Y|$ => $|X| = |Y|$.

Showing that Hall's Theorem conditions are satisfied is sufficient, as a matching saturating X (Y) will be perfect.

Consider an arbitrary $S \subseteq X$. The number m of edges connecting S to Y is $m = k|S|$ and those m edges are incident to $N(S)$.

The total number of edges incident to $N(S)$ is $k|N(S)|$. There is therefore $m \leq k|N(S)|$.

We thus obtained $k|S| = m \leq k|N(S)|$, satisfying Hall's Theorem sufficient condition $|S| \leq |N(S)|$. ■

Min-Max Dual Theorems

Can something be said on whether a matching is maximum when a complete matching does not exist?

Exploring all alternating paths to find whether or not there is an M-augmentation is hopeless.

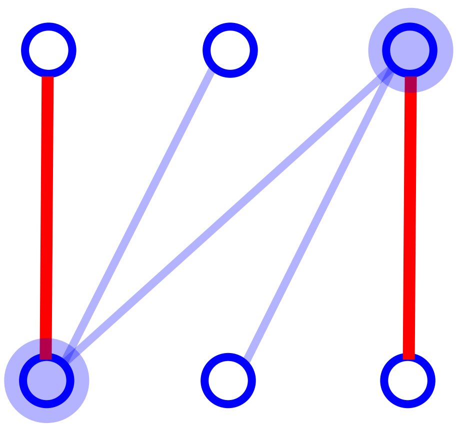
We rather consider a **dual problem** that answers it efficiently.

Definition. A **vertex cover** of G is a set S of vertices containing at least one vertex of all G's edges. We say that S's vertices **cover** G's edges.

No two edges in a matching are covered by a single vertex. The size of a cover is therefore **bounded below** by the **maximum matching** size.

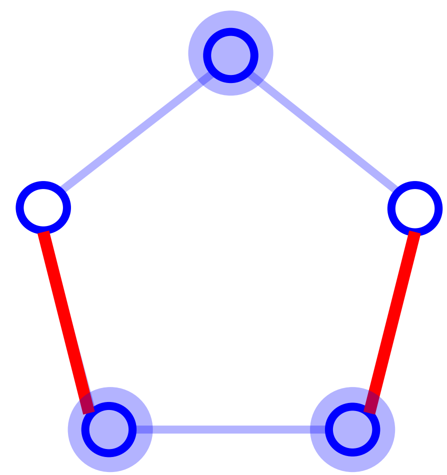
Exhibiting a cover and a matching of the same size will prove that both are **optimal**.

Each bipartite graph possesses such min-max equality, but general graphs not necessarily.



maximum matching=2

minimum cover=2,3



Theorem. (König 1931, Egerváry 1931) If $G[X,Y]$ is bipartite, the sizes of **maximum matching** and **minimum vertex cover** are equal.

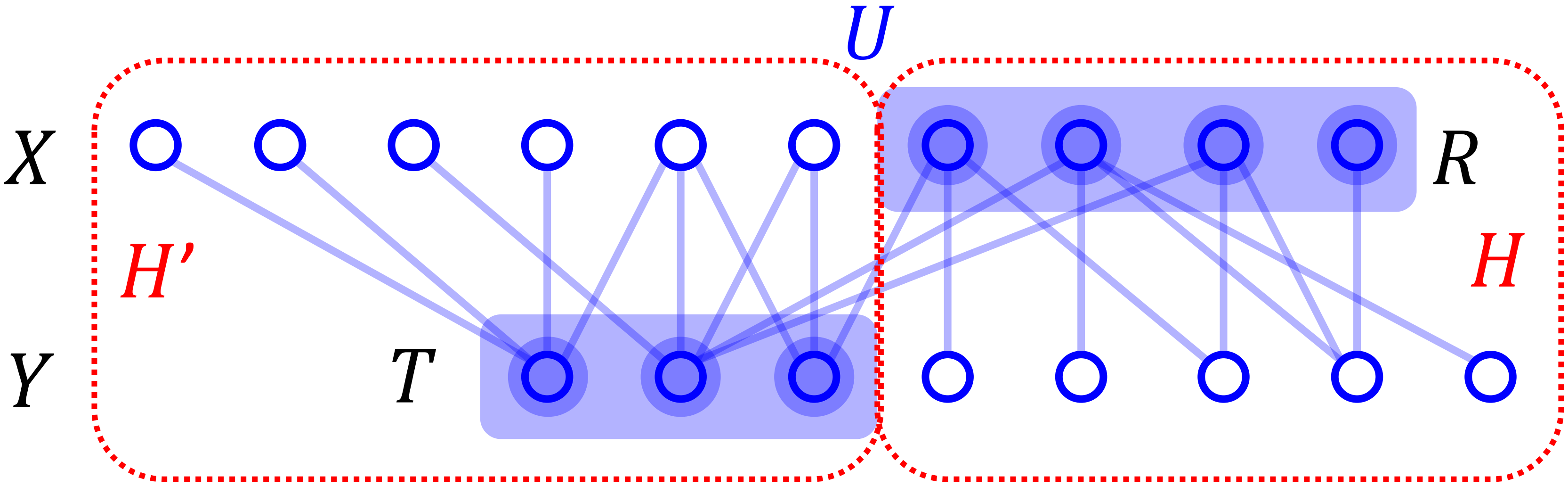
Proof. Let U be a G's vertex cover, and M a G's matching. There is always $|U| \geq |M|$.

Let U be a minimum cover. We subsequently construct a matching M from U such that $|U| = |M|$.

Let $R = U \cap X$ and $T = U \cap Y$. Two bipartite subgraphs H and H' are induced by $R \cup (Y - T)$ and $T \cup (X - R)$, respectively.

If we construct a complete matching in H of R into $Y-T$ and a complete matching in H' of T into $X-R$, their union will be a matching in G of size $|U|$, proving the theorem.

It is impossible to have an edge connecting $X-R$ with $Y-T$. Otherwise, U would not be a cover. Hence the matchings in H and H' are disjoint.

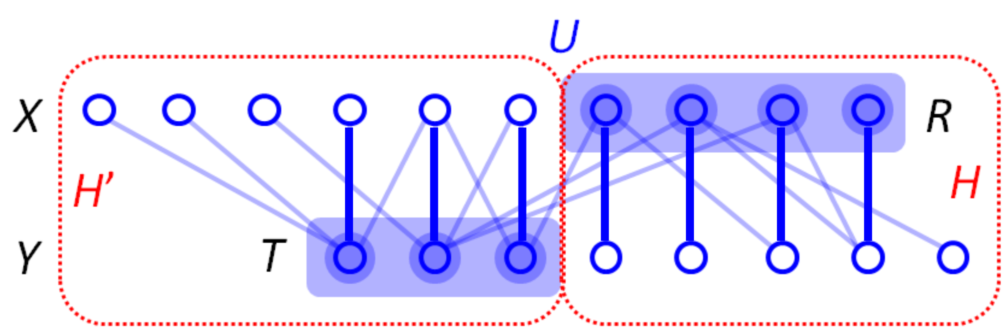


Showing that Hall's Theorem conditions are satisfied by H and H' will ensure that matchings saturating R and T exist.

Let $S \subseteq R$ and consider $N_H(S) \subseteq Y - T$. If $|N_H(S)| < |S|$ we could replace S by $N_H(S)$ in U and obtain a smaller vertex cover, contradicting U being minimum.

Therefore $|N_H(S)| \geq |S|$ and Hall's Theorem conditions hold in H. H has therefore a complete matching of R into $Y - T$.

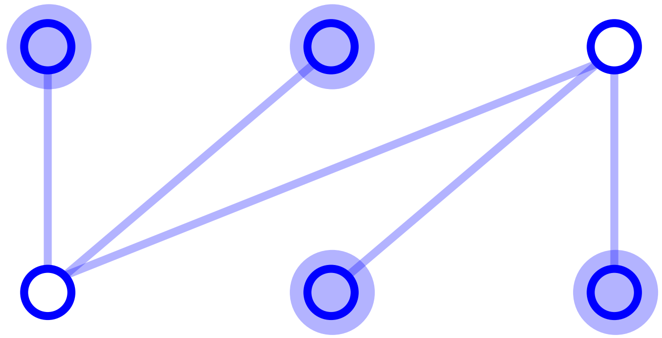
Same arguments hold for H'. ■



Independent Sets in Bipartite Graphs

Definition. The **independence number** $\alpha(G)$ of a graph G is the maximum size of an independent vertex set.

$\alpha(G)$ of a bipartite graph **does not** always equal the size of a partite set.



Definition. An **edge cover** is an edge set covering G's vertices.

Notation $\alpha(G)$: maximum size of independent set.
$\alpha'(G)$: maximum size of matching.
$\beta(G)$: minimum size of vertex cover.
$\beta'(G)$: minimum size of edge cover.

In this notation the König-Egerváry Theorem states that for every bipartite graph G, $\alpha'(G) = \beta(G)$.

Since there are no edges between the vertices of an independent set, the edge cover of the graph cannot be smaller, and therefore $\alpha(G) \leq \beta'(G)$.

We will also prove that for every bipartite graph G (without isolated vertices) $\alpha(G) = \beta'(G)$.

Lemma. $S \subseteq V(G)$ is an independent set iff $\bar{S}$ is a vertex cover, and hence $\alpha(G) + \beta(G) = n(G)$ ($n(G) := |V|$).

Proof. S independence => there are no edges within S, so $\bar{S}$ must cover all the edges. Conversely, $\bar{S}$ covers all the edges => no edges connecting two vertices of S. ■

Theorem. (Gallai 1959) If G has no isolated vertices, then $\alpha'(G) + \beta'(G) = n(G)$.

Proof. Let M be a maximum matching ($\alpha'(G) := |M|$). We can use it to construct an edge cover of G by adding an edge incident to each of the $n(G) - 2|M|$ unsaturated vertices, yielding edge cover of size $|M| + (n(G) - 2|M|) = n(G) - |M| = n(G) - \alpha'(G)$.

The smallest edge cover $\beta'(G)$ is a lower bound. Therefore, $n(G) - \alpha'(G) \geq \beta'(G)$.

Conversely, let L be a minimum edge cover ($\beta'(G)$: $= |L|$). L cannot contain cycles, nor paths of more than two edges. (why?)

L is therefore a collection of k **isolated star** subgraphs.

There are k vertices at star centers, anyway covered by the $n(G) - k$ peripheral. Thus $|L| = n(G) - k$.

The k isolated star subgraphs yield a k-size matching by arbitrarily choosing one edge per star.

A maximum matching cannot be smaller than k, thus $\alpha'(G) \geq k = n(G) - \beta'(G)$. All in all, $n(G) = \alpha'(G) + \beta'(G)$. ■

Corollary. (König 1916) If G is bipartite with no isolated vertices, $\alpha(G) = \beta'(G)$. (|maximum independent set| =|minimum edge cover|).

Proof. By the last lemma there is $\alpha(G) + \beta(G) = n(G)$. By Gallai Theorem there is $n(G) = \alpha'(G) + \beta'(G)$.

From König-Egerváry Theorem $\alpha'(G) = \beta(G)$ (|maximum matching|=|minimum vertex cover|), and $\alpha(G) = \beta'(G)$ follows. ■

Maximum Matching Algorithm

Augmentation-path characterization of maximum matching inspires an algorithm to find it.

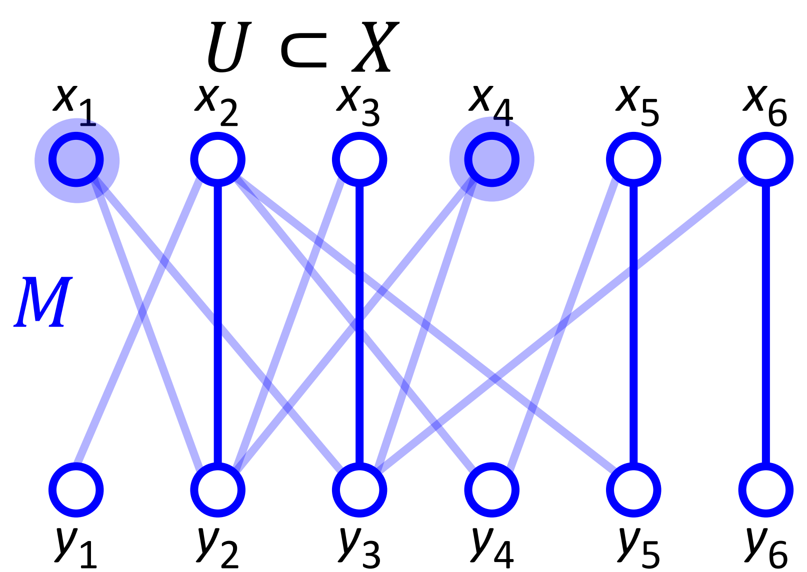
A matching is enlarged step-by-step, one edge at a time, by discovering an augmentation path.

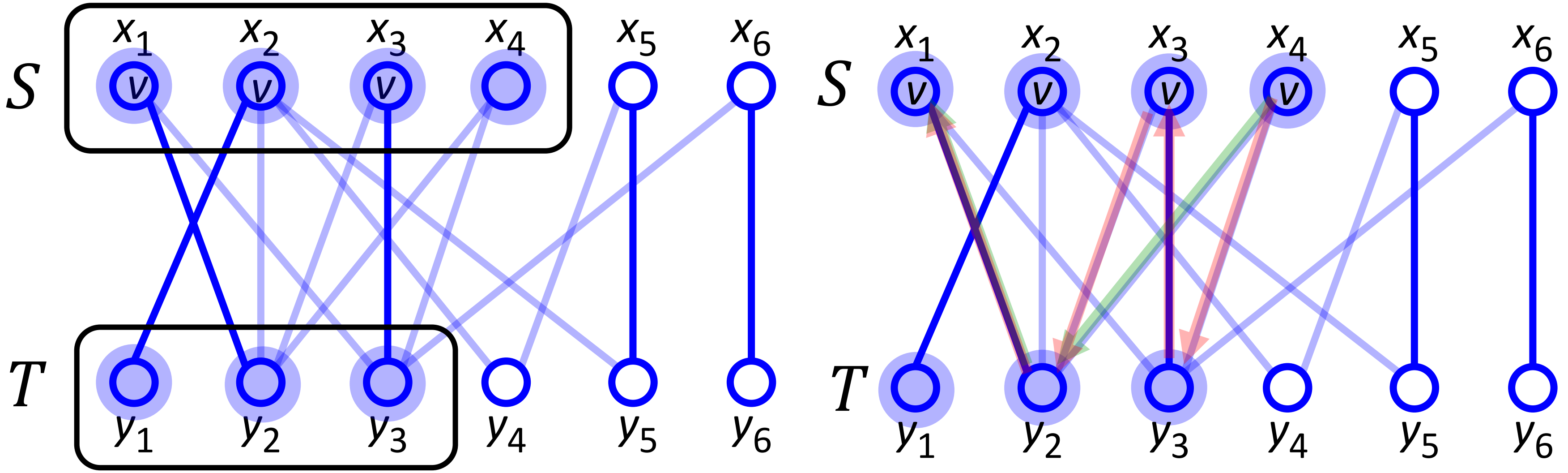
If an augmentation path is **not found**, there will be a vertex cover of same size as the current matching. König-Egerváry **Min-Max** Theorem ensures that the matching is maximum.

An iteration looks at M-unsaturated vertices only at **one partite** since an augmented path must have its two ends on distinct partite.

We search for all M-unsaturated vertices. Starting at an unsaturated vertex x, a tree of M-alternating paths rooted at x is implied.

Starting with zero matching, $\alpha'(G)$ applications of the Augmentation Path Algorithm produce a maximum matching.





No more unmarked in S. End of iteration.

A new iteration starts with y_4, where X and Y exchange roles. No augmentation will be found, while $S = \{y_1, y_4, y_5\}$ and $T = \{x_2, x_5\}$.

Algorithm (an iteration finding M-augmentation path). **Input**: $G[X, Y]$, matching M in G, all M-unsaturated vertices $U \subset X$.

Initialization: $S = U, T = \emptyset$.

Iteration: If all S is **marked** stop: M is a **maximum matching** and $T \cup (X - S)$ is a **minimum cover**.

Otherwise, select **unmarked** $x \in S$. Consider each $y \in N(S)$ such that $xy \notin M$. If y is unsaturated an M-ugmentation path from U to y exists. Augment M.

Otherwise, y is matched by M with some $w \in X$. In that case add y to T and w to S.

After exploring all edges incident to x, **mark** x and iterate. ■

Theorem. Repeated application of the Augmenting Path Algorithm to a bipartite graph produces matching and a cover of the same size.

Proof. Consider $T \cup (X - S)$ upon termination.

M-alternating path from U enters X only via M's edges, hence there is a matching between $S - U$ and T.

An M-alternating path traverses from $x \in S$ into T along any M-unsaturated edge, thus $N(x) \subset T$.

Since the algorithm marks all $x \in S$ before termination, there could not be an unsaturated edge connecting S to $Y - T$. $\boldsymbol{R = T \cup (X - S)}$ is therefore a **vertex cover**.

Upon termination T is saturated by M (otherwise M-augmentation occurs), hence $y \in T$ is M-matched to S.

Since $U \subseteq S$ contained all the M-unsaturated vertices, $X - S$ is M-saturated, but with edges not involved in T. M therefore involves at least $|T| + |X - S|$ edges.

Hence $\boldsymbol{|M| \geq |T| + |X - S| = |R|}$. Since **matching** size is **bounded above** by **covering** size, equality and optimality follow. ■

Weighted Bipartite Matching

Maximum matching in bipartite graphs generalizes to nonnegative weighted graphs. Missing edges are zero weighted, so $G = K_{n,n}$ is assumed.

Example. A farming company has n farms $X = \{x_1, \dots, x_n\}$ and n plants $Y = \{y_1, \dots, y_n\}$. The profit of processing x_i in y_j is $w_{ij} \geq 0$. Farms and plants should 1:1 matched.

The government will pay the company u_i to stop farm i production and v_j to stop plant j manufacturing.

If $u_i + v_j < w_{ij}$ the company will not take the offer and x_i and y_j will continue working.

What should the government offer to completely stop the farms and plants ?

It must offer $u_i + v_j \geq w_{ij}$ for all i, j. The government also wishes to minimize $\sum u_i + \sum v_j$.

Definitions. Given an $n \times n$ matrix A, a **transversal** is a selection of n entries, one for each row and one for each column.

Finding a transversal of A with maximum weight sum is called the **assignment problem**, a matrix formulation of the **maximum weighted matching** problem, where we seek a perfect matching M maximizing $w(M)$.

The labels $u = \{u_i\}$ and $v = \{v_j\}$ **cover** the weights $w = \{w_{ij}\}$ if $u_i + v_j \geq w_{ij}$ for all i, j.

The **minimum weighted cover** problem is to find a cover u, v minimizing the cost $c(u, v) = \sum u_i + \sum v_j$.

The maximum weighted matching and the minimum weighted cover problems are **dual**.

They generalize the bipartite maximum matching and minimum cover problem. (how ?)

The edges are assigned with weight from {0, 1}, and the cover is restricted to use only integral labels from {0, 1}. Vertices receiving 1 form the cover.

Lemma. If M is a complete (perfect) matching in a bipartite graph G and u, v is a cover, $c(u, v) \geq w(M)$.

Furthermore, $c(u, v) = w(M)$ iff M consists of edges $x_i y_j$ such that $u_i + v_j = w_{ij}$. M is then a **maximum weight** matching and u, v is a **minimum weight** cover.

Proof. Since the edges of M are **disjoint**, it follows from the constraints $u_i + v_j \geq w_{ij}$ that summation over all *M*'s edges yields $c(u, v) \geq w(M)$.

If $c(u, v) = w(M)$ equality $u_i + v_j = w_{ij}$ must hold for each of the n summand.

Finally, since $w(M)$ is bounded by $c(u, v)$, equality implies that both must be optimal. ■

Weighted Bipartite Matching Algorithm

The relation between maximum weighted matching and edge covered by equalities lends itself to an algorithm, named the **Hungarian Algorithm**.

It combines M-augmentation path with cover trimming.

Denote by $G_{u,v}$ the subgraph of $K_{n,n}$ spanned by the edges $x_i y_j$ satisfying $u_i + v_j = w_{ij}$.

The algorithm ensures that if $G_{u,v}$ has a perfect matching (in G), its weight is $\sum u_i + \sum v_j$ and by the lemma both matching and cover are optimal.

Otherwise, the algorithm modifies the cover.

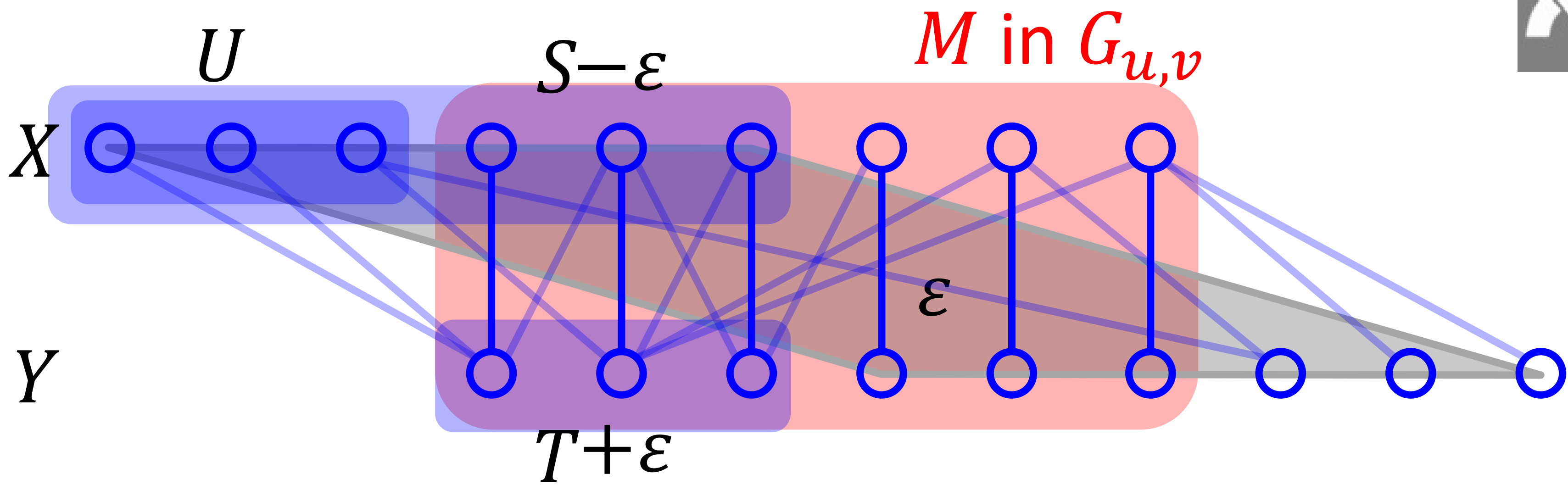
Algorithm. (Kuhn 1955, Munkres 1957)
Input: Bipartition $G[X, Y]$ and weights of $K_{n,n}$.

Idea: maintain a cover u, v, iteratively reducing its cost, until the equality graph $G_{u,v}$ has a perfect matching.

Initialization: Define a feasible labeling $u_i = \max_j w_{ij}$, and $v_j = 0$. Find a maximum matching M in $G_{u,v}$ (apply path augmentation to $G_{u,v}$).

Iteration: If M is perfect in $G[X, Y]$ stop. M is a maximum weight matching by the lemma.

Else, let $U \subset X$ be the M-unsaturated in X and $S \subseteq X$, and $T \subset Y$ be reached from U by M-alternating paths.

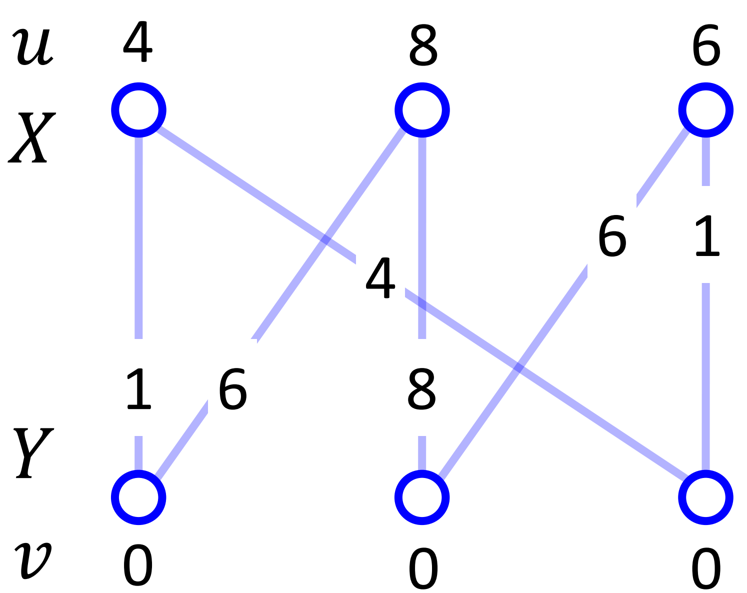


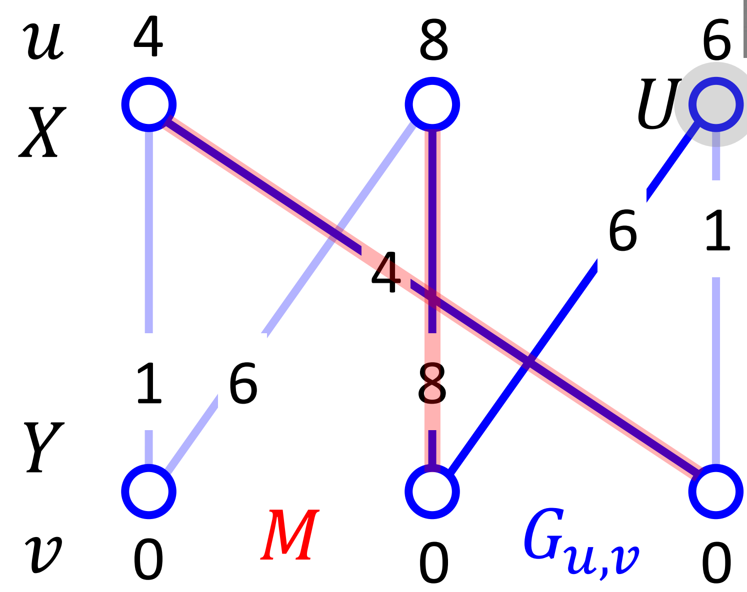
Let $\varepsilon = \min\{u_i + v_j - w_{ij} \mid x_i \in S, y_j \in Y - T\}$

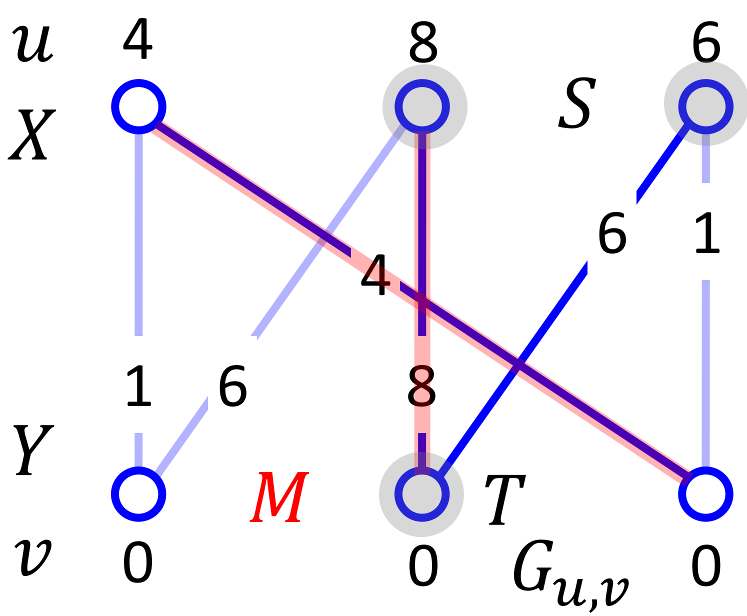
Decrease u_i by ε for all $x_i \in S$ and increase v_j by ε for all $y_j \in T$.

Derive a new equality graph $G'_{u,v}$. If it contains an M-augmentation path, replace M by a maximum matching in $G'_{u,v}$.

Iterate anyway. ■

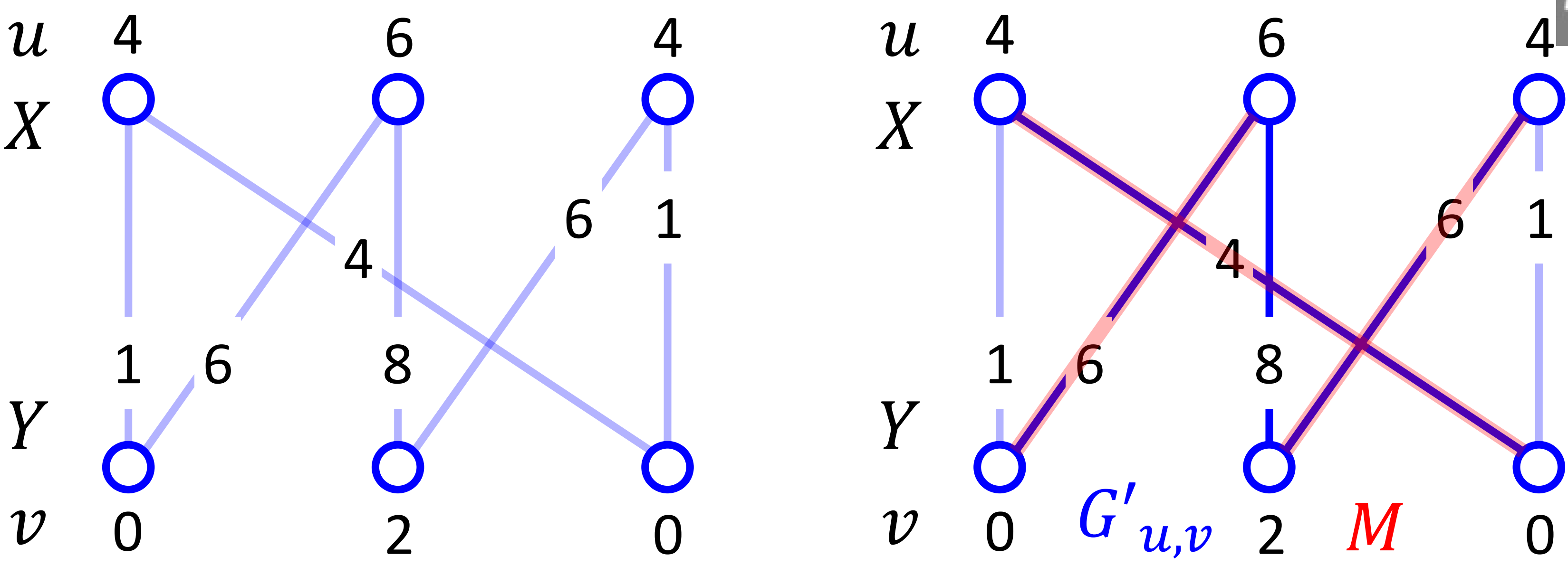






Minimum surplus from S to $Y - T$:

$$\varepsilon = \min\{8 - 6, 6 - 1\} = 2$$

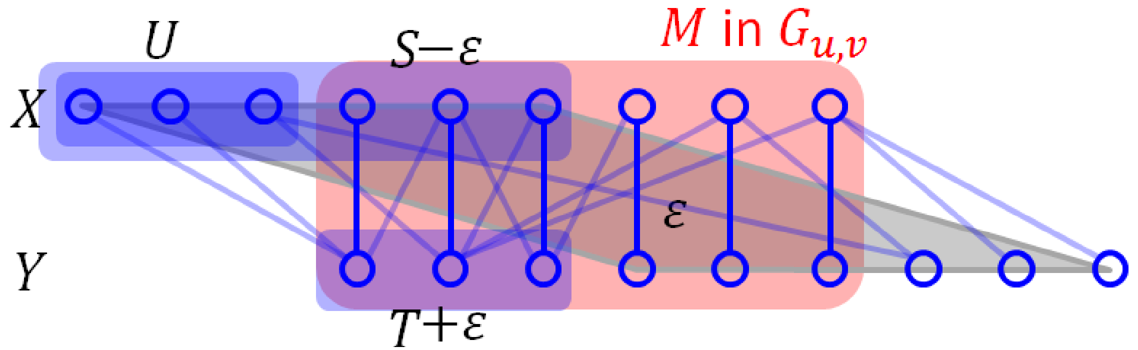


M which is a maximum in $G'_{u,v}$ is a perfect matching in G and therefore it is maximum weight matching.

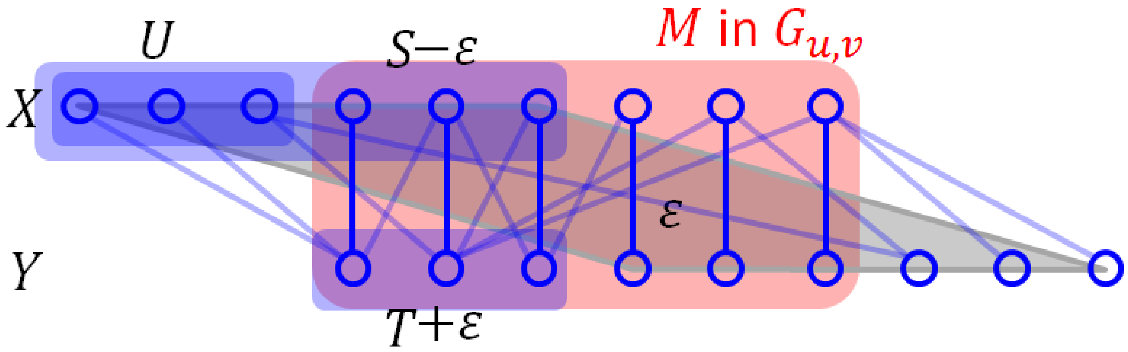
To validate, the total edge weight is 16, same as the total cover.

Theorem. The Hungarian Algorithm finds a maximum weight matching and a minimum cost cover.

Proof. The algorithm begins with a cover, each iteration produces a cover, and termination occurs only when the equality graph $G_{u,v}$ has a perfect matching in G.



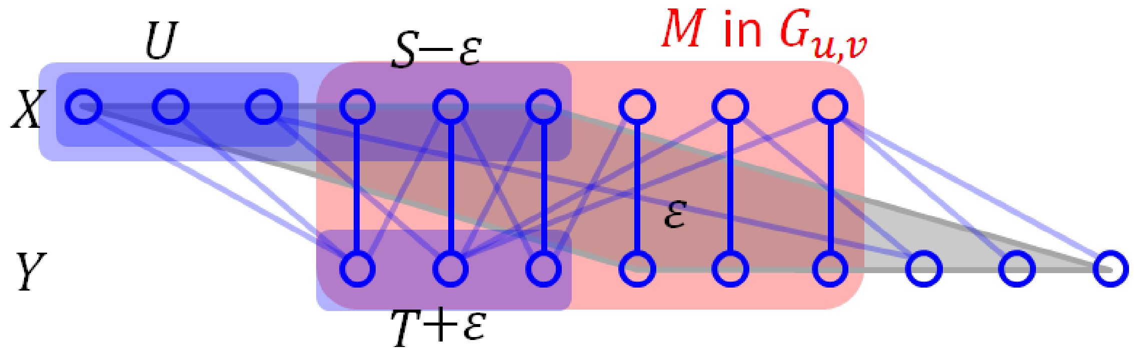
Consider the numbers u', v', obtained from the cover u, v, after decreasing S and increasing T by ε. If $x_i \in S$ and $y_j \in T$, then $u'_i + v'_j = u_i + v_j$ and cover holds.



If $x_i \in X - S$ and $y_j \in Y - T$, then $u'_i + v'_j = u_i + v_j$ and cover holds.

If $x_i \in X - S$ and $y_j \in T$, then $u'_i + v'_j = u_i + v_j + \varepsilon$, hence cover holds.

Finally, if $x_i \in S$ and $y_j \in Y - T$, then $u'_i + v'_j = u_i + v_j - \varepsilon$. Since ε was the smallest surplus from S to $Y - T$, cover holds.



The termination condition ensures that optimum is achieved. It is required therefore to show that termination occurs after a finite number of iterations.

First, $|M|$ never decreases since $G_{u,v} \subset G'_{u,v}$. If $|M|$ increases, great. If not, then $|T|$ increases in $G'_{u,v}$.

That follows from the addition of a new cover-equal edge between S and $Y-T$, traversed in $G'_{u,v}$ by an M-alternating path emanating from U (M-unsaturated). ■

Stable Matching

Preferences are optimized Instead of total weight.

A matching of n men and n women is **stable** if there is no man-woman pair (x, a) such that x and a prefer each other over their current partners.

Otherwise the matching is **unstable**; x and a will leave their current partners and will switch to each other.

Men: $\{x, y, z, y\}$	Women: $\{a, b, c, d\}$
x: $a > b > c > d$	a: $z > x > y > w$
y: $a > c > b > d$	b: $y > w > x > z$
z: $c > d > a > b$	c: $w > x > y > z$
w: $c > b > a > d$	d: $x > y > z > w$

$\{xa, yb, zd, wc\}$ is stable.

Gale and Shapley proved that a stable matching always exists and can be found by a simple algorithm.

Algorithm. (Gale-Shapley **Proposal Algorithm**).
Input: Preference ranking by each of n men and n women.

Idea: produce stable matching using proposals while tracking past proposals and rejections.

Iteration: Each unmatched man proposes to the highest woman on his list which has not yet rejected him.

If each woman receives one proposal stop. a **stable matching** is obtained.

Otherwise, at least one woman receives at least two proposals. Every such woman rejects all but the **highest** on her list, to which she says "maybe". ■

Theorem. (Gale-Shapley 1962) The Proposal Algorithm produces stable matching.

Proof. The algorithm terminates (with some matching) since at each nonterminal iteration at least one woman rejects a man, reducing the list of n^2 potential mates.

Observation: the proposals sequence made by a man is non increasing in his preference list, whereas the list of "maybe" said by a woman is non decreasing in her list.

(Repeated proposals by a man to the same woman and repeated “maybe” answers are possible, until rejected or assigned.)

If matching is unstable, there is (x, b) and (y, a) mates, where 1: x prefers a over b and 2: a prefers x over y.

1: Since on its preference list $a > b$, x proposed to a before it proposed to b, a time where x must have already been rejected by a.

2: By the observation, the “maybe” answer sequence made by a is non decreasing in its preferences. Since on a’s list $y < x$, x could never propose to a, hence a contradiction to 1. ■

Question: Which gender is happier using Gale-Shapley Algorithm? (The algorithm is asymmetric.)

When the first choice of all men are distinct, they all get their highest possible preference ☺, whereas the women are stuck with whomever proposed ☹.

The precise statement of "the men are happier" is that if we switch the role of men and women, each woman winds up happy and each man winds up unhappy at least as in the original proposal algorithm. (homework)

If women propose to men they get $\{xd, yb, za, wc\}$, which are their first choices .

Of all stable matching, men are happiest by the male-proposal algorithm whereas women are happiest by the female-proposal algorithm. (homework)

Study the case where all men and all women have same preference lists. (homework)

The algorithm can be used for assignments of new graduates of medicine schools to hospitals.

Who is happier, young doctors or hospitals? Hospitals are happier since they run hospital-proposal.

Hospitals started using it on early 50’s to avoid chaos, ten years before Gale-Shapley algorithm was proved.

Matching in Arbitrary Graphs

We shall establish a lower bound on the uncovered vertices of a maximum matching M in a graph G.

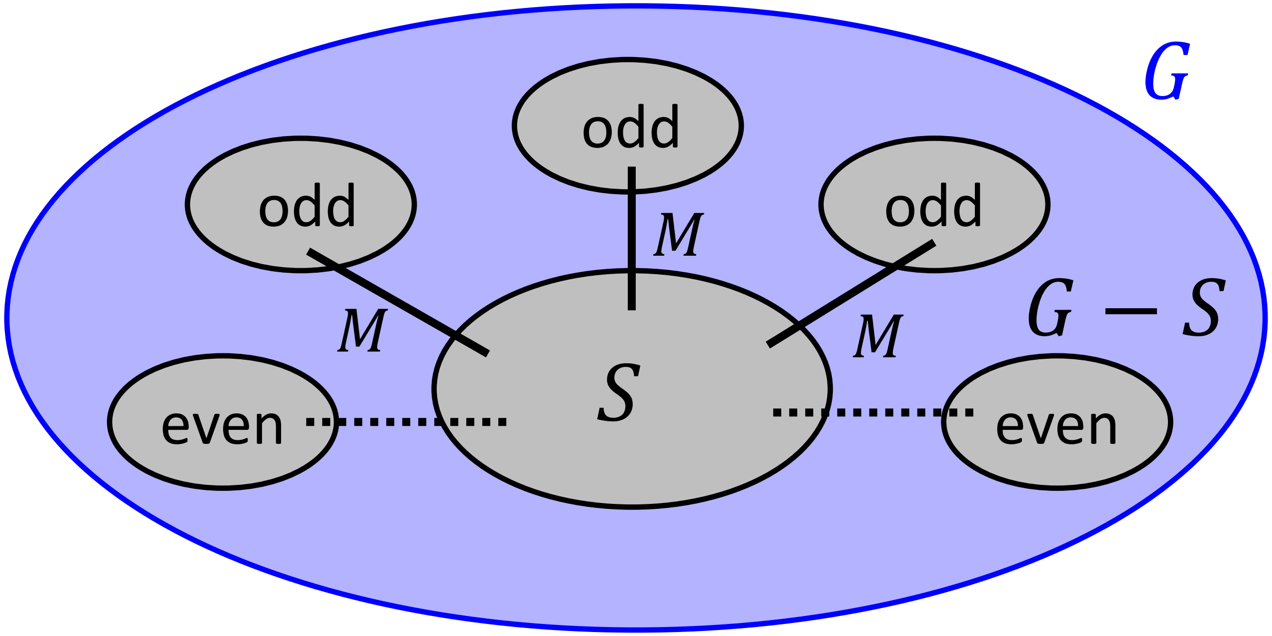
Definition. An **odd component** of a graph is a component of odd order (odd number of vertices).

$o(G)$ is the number of odd components of a graph G.

If M is a matching in G and U is the uncovered vertices, each odd component of G must include at least one vertex not covered by M, hence $|U| \geq o(G)$.

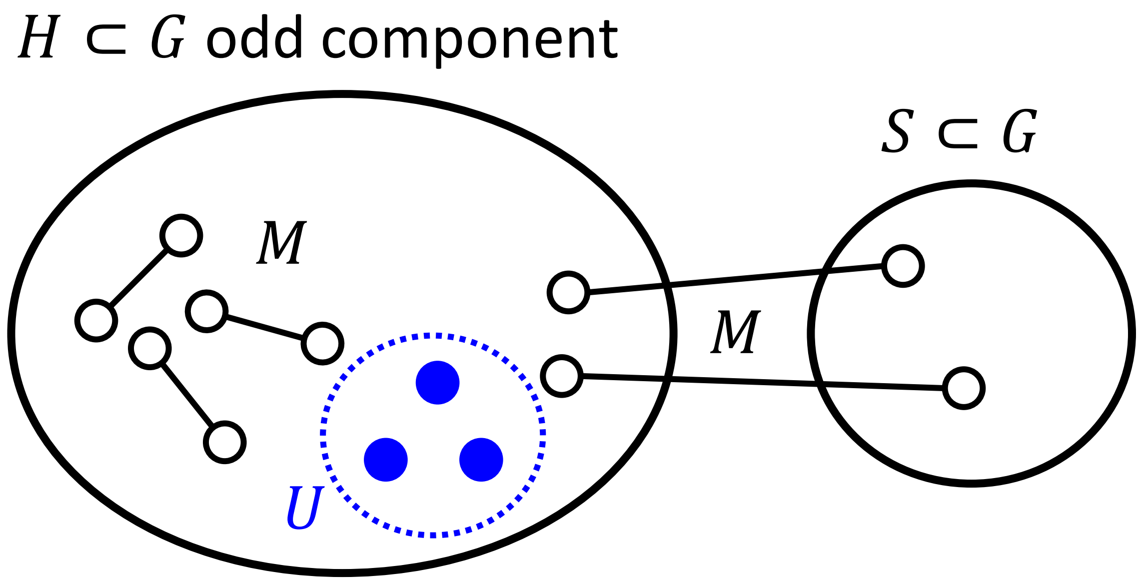
This inequality can be extended to all **induced subgraphs** of G.

Let $S \subset V(G)$, and let H be an **odd component** of $G - S$. If H is fully covered by M (all H's vertices touched), there must be at least one $v \in H$ matching a vertex of S.

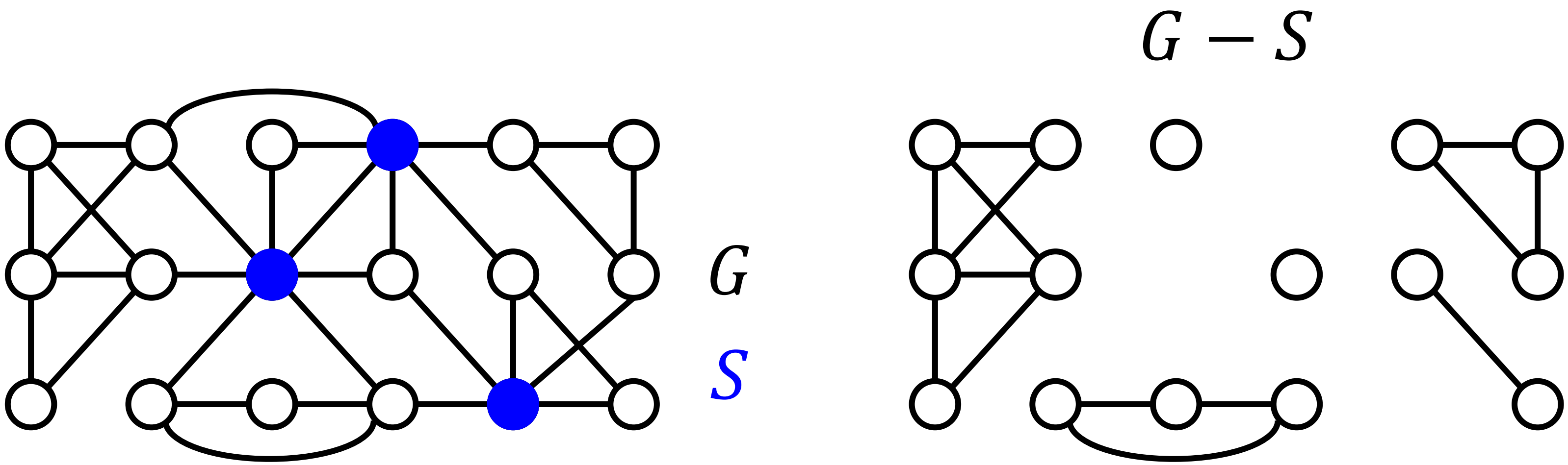


At most $|S|$ vertices of $G - S$ can be matched by M with those of S.

Consider the uncovered vertices U of a matching M in G.



At least $o(G-S)-|S|$ odd components must have a vertex not covered by M, hence $|U| \geq o(G-S)-|S|$, for any $S \subset V(G)$.



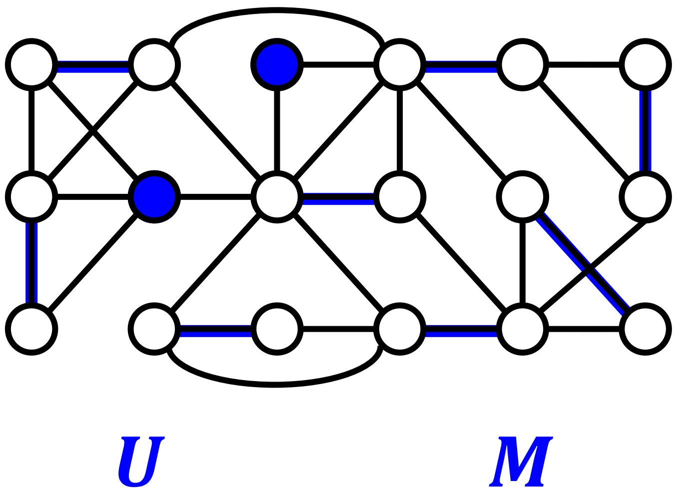
Does G have a perfect matching ?

$o(G - S) = 5$, whereas $|S| = 3$, hence $|U| \geq 2$.

Claim. If it happens that for some matching M and $B \subset V(G)$ there is $|U| = o(G - B) - |B|$, then M is a maximum matching. (homework)

Such B is called a **barrier** of G and is a certificate that M is maximum.

$|U| \geq 2$



M is maximum since $|U| = 2$.

The empty set is trivially a barrier of any graph possessing a **perfect matching** since $|U| = o(G) = 0$.

Any single vertex is also a barrier of any graph possessing a **perfect matching**. (why?)

The empty set is a barrier of a graph for which a deletion of one vertex results in a subgraph possessing a **perfect matching**. (why?)

The union of barriers of the components of a graph is a barrier of the graph. (homework)

Any minimum covering of a bipartite graph is a barrier. (homework)

Definition: A **factor** of G is a spanning subgraph of G.

Definition: A ***k*-factor** is a ***k*-regular** (all vertices have degree k) spanning subgraph. A **perfect matching** is **1-factor**.

Theorem. (Tutte's 1-Factor Theorem 1947) A graph G has 1-factor iff $o(G-S) \leq |S|$ for every $S \subseteq V(G)$.

Proof (Lovász 1975). (Only if) Let G have 1-factor (perfect matching) and $S \subseteq V(G)$. 1-factor $\Rightarrow o(G-S) \leq |S|$ was shown before.

The proof of the opposite direction is more complex. Tutte's condition is preserved under edge addition, namely, if $o(G-S) \leq |S|$, so it is for $G' = G + e$.

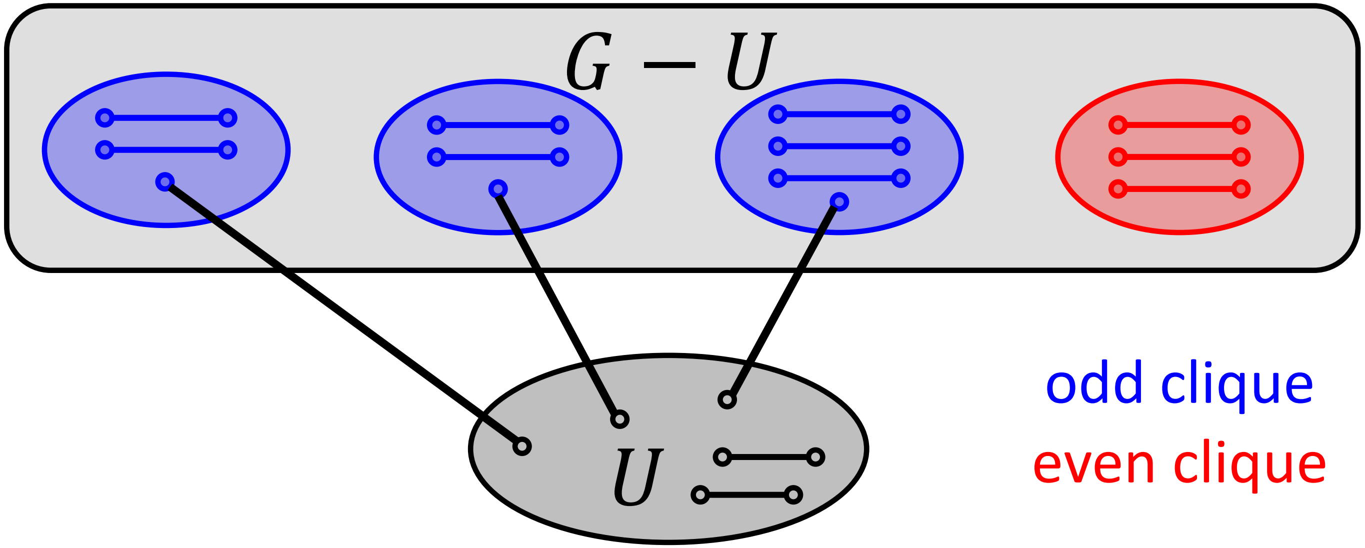
That follows since edge addition may merge two components into one, hence $o(G' - S) \leq o(G - S) \leq |S|$.

Proof plan. We will consider a graph G possessing Tutte's condition and assume in contrary that it has no 1-factor, but the addition of any edge obtains 1-factor.

We then add two edge e, f and construct a 1-factor in $G' = G + \{e, f\}$. We then remove e, f and show the existence of 1-factor in G, hence a contradiction.

$n(G)$ must be even. That follows by taking $S = \emptyset$, so $o(G - S) \leq 0$, hence no odd component could exist.

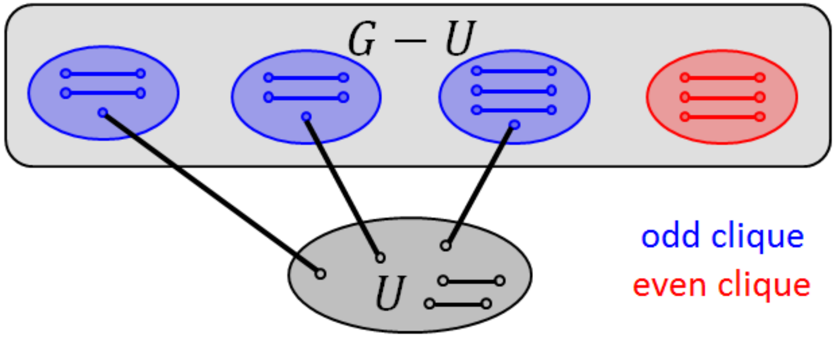
Let $U \subseteq V(G)$ be such that $v \in U$ is connected to all G's vertices and suppose that $G - U$ consists of disjoint **cliques**.



odd clique

even clique

$G - U$ vertices are arbitrarily paired up, with the leftover residing in the odd components. Since $o(G - U) \leq |U|$, the leftover ones matched to any of U.



U is a clique of its own, hence all its rest vertices (even) are paired up and 1-factor in G exists.

We must therefore consider for the contradiction establishment that $G-U$ is not made all of cliques.

There must be non-adjacent vertices $x, z \in G-U$ sharing a common vertex y (otherwise x, z are in a clique, or $G-U$ is an independent set).

Since $y \notin U$, there is $w \in G-U$, non adjacent to y.

Recall that G was chosen to be maximal not having 1-factor, such that the addition of any edge will turn it to possess 1-factor.

Let M_1 and M_2 be the 1-factors in $G + xz$ and $G + yw$, respectively. By G maximality $xz \in M_1$ and $yw \in M_2$.

It suffices to show that $M_1 \cup M_2$ has 1-factor avoiding xz and yw, in contradiction with the maximality of G.

Consider G with the edges of $F = M_1 \Delta M_2$. There is $xz, yw \in F$.

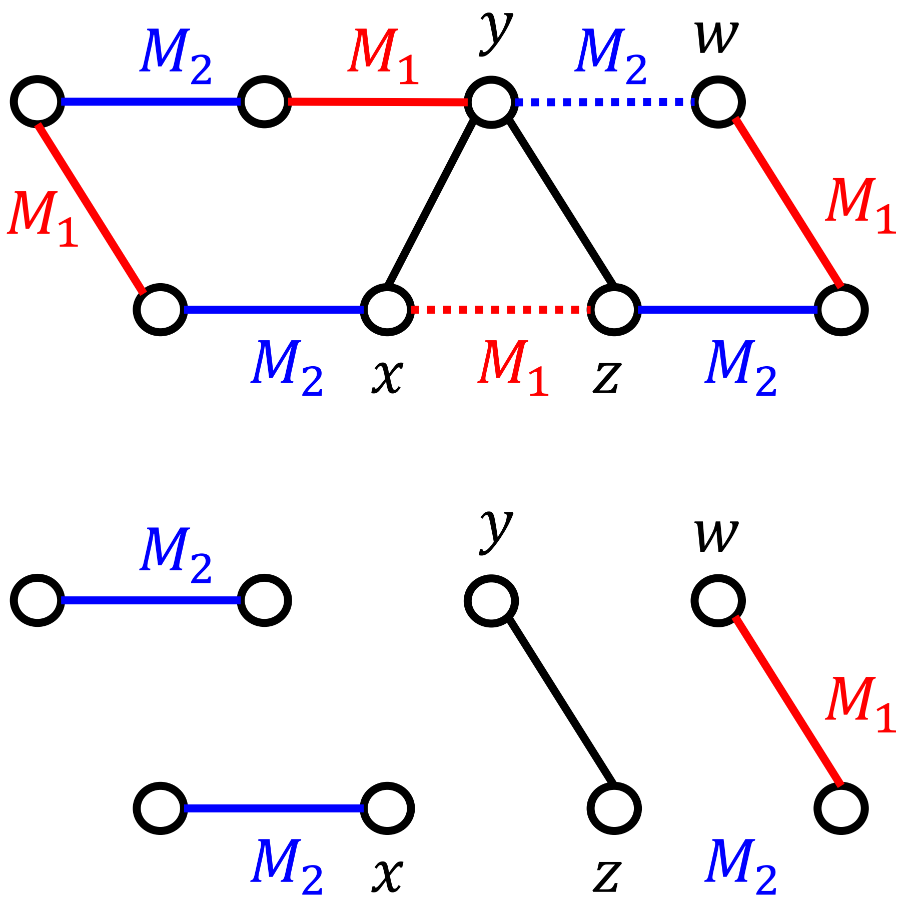
Since the degree of any G's vertex in M_1 and M_2 is exactly one (perfect matchings), F has only isolated vertices and disjoint even cycles.

Let C be the cycle in F, $xz \in C$. If $yw \notin C$, a 1-factor is established by $(M_2 \cap C) \cup (M_1 \backslash C)$, avoiding both xz and yw.

If both $xz, yw \in C$, the following cycle occurs.

Here is a matching of $V(C)$ avoiding both xz and yw. Outside C either M_1 or M_2 edges are used, yielding perfect matching in G.

■

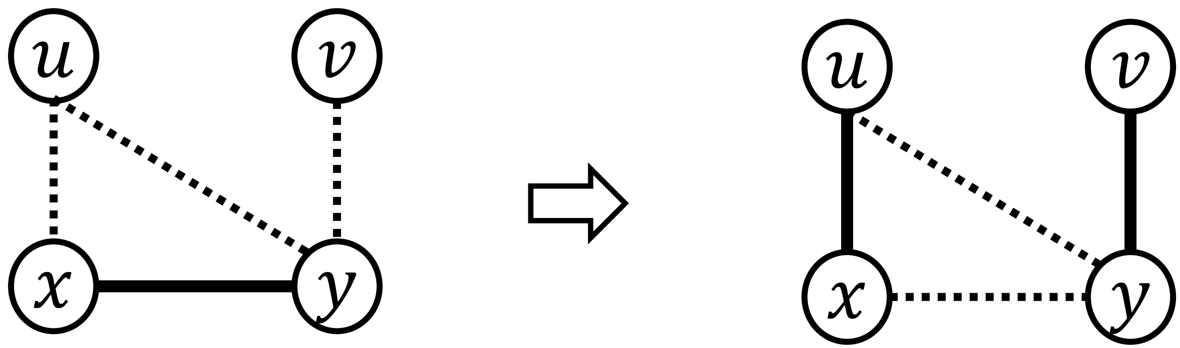


Example. Let $G(V,E)$ be a simple graph (no parallel edges and loops) of $2n$ vertices. Let the degree $d(v) \geq n$, $\forall v \in V$. Show that G has a perfect matching.

Solution. If G has no perfect matching, let F be a largest (maximum) possible matching.

Let $(x,y) \in F$. Since F is not perfect and $|V| = 2n$, there are at least two unmatched vertices $\{u,v\}$ and edge (u,v) does not exist.

Consider how $\{x,y\}$ and $\{u,v\}$ can be connected. Assume there are at least 3 edges involved.



There are 2 independent edges which can be used for matching if (x, y) is removed, contradicting that F is a largest matching.

Consequently, $\{u, v\}$ cannot be both connected to any of the vertices involved in F, which number is at most $2n - 2$. Consequently, $d(u) + d(v) \leq 2n - 2$.

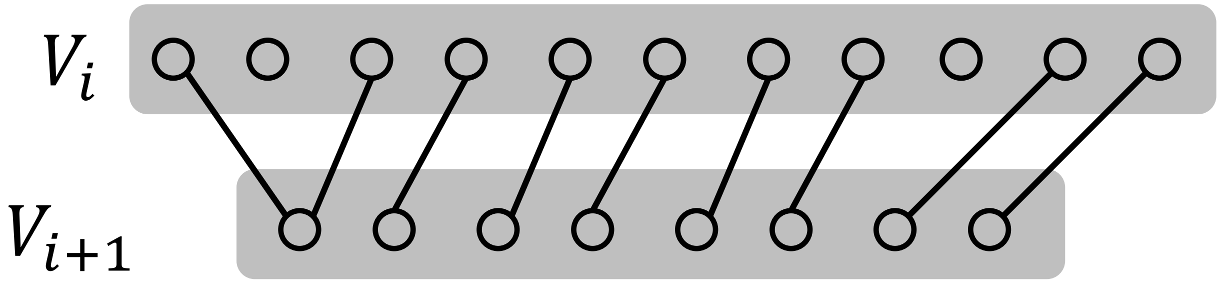
But $d(u) + d(v) \geq 2n$, hence a contradiction. ■

Problem Denote by $\alpha(G)$ the size of the largest independent set of G. Show that the vertices of a graph $G(V, E)$ can be covered by no more than $\alpha(G)$ vertex-disjoint paths.

Proof: Let V_1 be a maximum independent set of G, and let V_{i+1} be the maximum independent set of $G - V_1 - V_2 - \cdots - V_i$.

There is $|V_{i+1}| \leq |V_i|$ by construction.

Since V_i and V_{i+1} are independent sets, for any $e(x, y) \in G[V_i \cup V_{i+1}]$ there is $x \in V_i$ and $y \in V_{i+1}$. Hence $G[V_i \cup V_{i+1}]$ is bipartite, denoted $G[V_i, V_{i+1}]$.



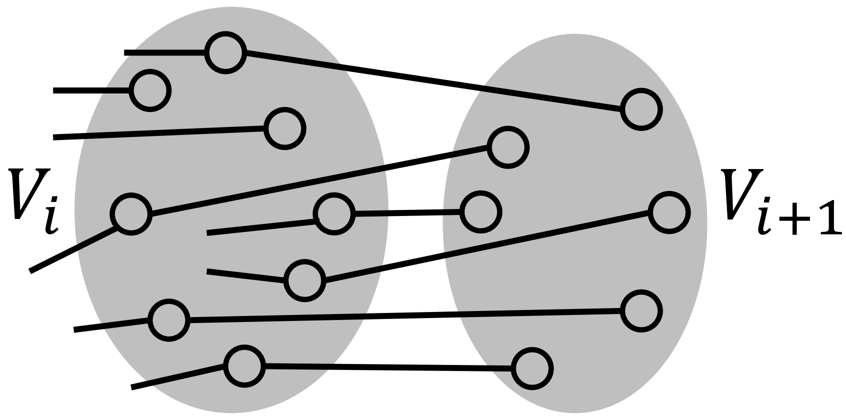
We show that the minimum vertex cover satisfies $\beta(G[V_i, V_{i+1}]) = |V_{i+1}|$.

Firstly, for any $v \in V_i$ there is $d(v) \leq 1$. Otherwise V_i would not be maximal by its choice since it could be enlarged by replacing $v \in V_i$ by few vertices of V_{i+1}.

Hence, the minimum vertex cover may consist of V_{i+1} vertices alone.

It must include all the vertices of V_{i+1}, as otherwise V_i could be enlarged, hence not maximal by its choice. Consequently $\beta(G[V_i, V_{i+1}]) = |V_{i+1}|$.

By König's theorem there is $\beta(G[V_i \cup V_{i+1}]) = \alpha'(G[V_i \cup V_{i+1}])$. Hence there is a matching F_{i+1} of V_{i+1} into V_i.



$F_2 \cup F_3 \cup \cdots$ consists of $|V_2|$ vertex-disjoint paths covering $V(G)$, except $|V_1| - |V_2|$ vertices of V_1.

Taking these as one-point paths, we obtain $|V_1| = \alpha(G)$ vertex-disjoint paths covering $V(G)$. ■