MCS.T419 Stochastic Differential Equations
Solutions to Assignment #2

Problem 1.

$$X_t = \begin{bmatrix} X_0^1 \cos t - X_0^2 \sin t - \sigma^2 \int_0^t \sin(t-s) dW_s \\ X_0^1 \sin t + X_0^2 \cos t + \sigma^2 \int_0^t \cos(t-s) dW_s \end{bmatrix}.$$

Problem 2.

$$X_t = \sinh(t + W_t + \operatorname{arcsinh}(X_0)).$$

Problem 3. Omitted.

Problem 4. Omitted. You may find that the nonparametric methods need a very large number of samples.

Problem 5. First, it should be remarked that the initial wealth X_0 is implicitly assumed to be a positive constant. Then, our task is to check the condition (i) in Theorem 4.11. In particular, it is sufficient to show that $\mathbb{E}[\sup_{0\le t\le T} e^{Y_t^a}] < \infty$ for any $a \in \mathbb{R}$. To this end, observe

$$e^{Y_t^a} \le C Z_t^2$$

where

$$C = \exp\left(\left(\frac{1}{4}q^2\sigma^2 a^2 - \frac{1}{2}q\sigma^2 a^2 + qr + q(b-r)a\right) T\right),$$
$$Z_t = \exp\left(\frac{q\sigma a}{2} W_t - \frac{1}{2}\left(\frac{q\sigma a}{2}\right)^2 t\right).$$

Applying Doob's maximal inequality (Theorem 1.33), we obtain

$$\mathbb{E} \sup_{0\le t\le T} Y_t^a \le C\mathbb{E} \sup_{0\le t\le T} Z_t^2 \le 2C\mathbb{E}Z_T^2 < \infty.$$

Problem 6. Fix $(t,x) \in [0,1) \times \mathbb{R}$, let $T' \in (t,1)$ and consider the stopping time

$$\tau_n := \inf\{s \in [t,T'] : |\partial_x u(s, W_s^{t.x})| \ge n\} \wedge T',$$

where $W_s^{t,x} = x + W_s - W_t$. Further, put $Y_s = e^{i\lambda \int_t^s g(r,W_r^{t,x})dr + i\mu \int_t^s h(r,W_r^{t,x})dr}$. Then, by Itô formula and the assumption,

$$\mathbb{E}[u(\tau_n, W_{\tau_n}^{t,x}) Y_{\tau_n}] = u(t,x).$$

Letting $n \to \infty$ and then $T' \to 1$, we get the required result.

Note. The equation for u should be

$$\partial_t u(t,x) + \frac{1}{2}\partial_{xx}^2 u(t,x) + \left(i\lambda g(t,x) - \frac{1}{2}\mu^2 h(t,x)^2\right) u(t,x) + i\mu h(t,x) \partial_x u(t,x) = 0$$

rather than the one described in the statement. Sorry for this mistake.

Problem 7. First solve

$$\sum_{j=1}^{n} \frac{(X_j - X_{j-1} - bX_{j-1}\Delta)^2}{X_{j-1}}$$

with $\Delta = 1$ to determine $\hat{b}$, and then put

$$\hat{\sigma}^2 = \frac{1}{n}\left(\sum_{j=1}^{n} \frac{(X_j - X_{j-1})^2}{X_{j-1}} - \frac{(\sum_{j=1}^{n}(X_j - X_{j-1}))^2}{\sum_{j=1}^{n} X_{j-1}}\right).$$

The resulting estimated parameters are $\hat{b} = 0.0361$ and $\hat{\sigma} = 0.7806$.

Problem 8. The kernel-based collocation method as in the numerical examples in p.99-100 can be available. The details are omitted.