MCS.T419 Stochastic Differential Equations
Solutions to Assignment #1

Problem 1.
(i). S_0 is assumed to be constant.

$$\sigma(S_1) = \{\{S_1 \in A\} : A \in \mathcal{B}(\mathbb{R})\} = \{\Omega, \emptyset, \{D_1 = u\}, \{D_1 = d\}\}.$$

(ii). By Proposition 1.7,

$$\mathbb{E}[S_2|S_1] = \mathbb{E}[D_2 S_1|S_1] = S_1\mathbb{E}[D_2] = S_1(up + d(1-p)).$$

Problem 2. Let $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R})$. Put

$$N = \{\lim_{n\to\infty} X_t^{(n)} \neq X_t \text{ or } \lim_{n\to\infty} X_t^{(n)} \text{ does not exist}\}.$$

Then, since $N^c, \{X_t \in A\} \cap N \in \mathcal{F}_0$ and $\limsup_{n\to\infty} X_t^{(n)}$ is $\mathcal{F}_t$-measurable, we see

$$\begin{aligned}\{X_t \in A\} &= (\{X_t \in A\} \cap N) \cup (\{X_t \in A\} \cap N^c) \\ &= (\{X_t \in A\} \cap N) \cup (\{\limsup_{n\to\infty} X_t^{(n)} \in A\} \cap N^c) \in \mathcal{F}_t.\end{aligned}$$

Problem 3. The function $g(x) := |x|^p$ is convex, and so we can apply Proposition 1.7 (xi) to obverse

$$\mathbb{E}[|M_t|^p|\mathcal{F}_s] = \mathbb{E}[g(M_t)|\mathcal{F}_s] \geq g(\mathbb{E}[M_t|\mathcal{F}_s]) = g(M_s) = |M_s|^p$$

for $t \geq s$.

Problem 4. First notice that an a.s. continuous process $\{X_t\}$ with $X_0 = 0$ is a BM if and only if

$$\mathbb{E}\left[\exp\left(\sqrt{-1}\sum_{k=1}^m \xi_k(X_{t_k} - X_{t_{k-1}}\right)\right] = e^{(1/2)\sum_{k=1}^m \xi_k^2(t_k - t_{k-1})}$$

for any $\xi_1, \ldots, \xi_m \in \mathbb{R}$, $0 \leq t_0 \leq t_1 \leq \cdots t_m$, and $m \in \mathbb{N}$.

(i). $X_t := -W_t$ satisfies

$$\begin{aligned}\mathbb{E}\left[\exp\left(\sqrt{-1}\sum_{k=1}^m \xi_k(X_{t_k} - X_{t_{k-1}}\right)\right] &= \mathbb{E}\left[\exp\left(\sqrt{-1}\sum_{k=1}^m (-\xi_k)(W_{t_k} - W_{t_{k-1}})\right)\right] \\ &= e^{(1/2)\sum_{k=1}^m \xi_k^2(t_k - t_{k-1})}.\end{aligned}$$

(ii). $X_t := W_{t+s} - W_s$ satisfies

$$\begin{aligned}\mathbb{E}\left[\exp\left(\sqrt{-1}\sum_{k=1}^m \xi_k(X_{t_k} - X_{t_{k-1}}\right)\right] &= \mathbb{E}\left[\exp\left(\sqrt{-1}\sum_{k=1}^m \xi_k(W_{t_k+s} - W_{t_{k-1}+s})\right)\right] \\ &= e^{(1/2)\sum_{k=1}^m \xi_k^2(t_k - t_{k-1})}.\end{aligned}$$

(iii). $X_t := cW_{t/c^2}$ satisfies

$$\begin{aligned}\mathbb{E}\left[\exp\left(\sqrt{-1}\sum_{k=1}^{m}\xi_k(X_{t_k}-X_{t_{k-1}})\right)\right] &= \mathbb{E}\left[\exp\left(\sqrt{-1}\sum_{k=1}^{m}\xi_k c(W_{t_k/c^2}-W_{t_{k-1}/c^2})\right)\right]\\ &= e^{(1/2)\sum_{k=1}^{m}c^2\xi_k^2(t_k/c^2-t_{k-1}/c^2)}\\ &= e^{(1/2)\sum_{k=1}^{m}\xi_k^2(t_k-t_{k-1})}.\end{aligned}$$

Problem 5. By Proposition 1.7, for $t \geq s$,

$$\begin{aligned}\mathbb{E}[M_t|\mathcal{F}_s] &= \mathbb{E}[N_t - N_s + N_s - \lambda t|\mathcal{F}_s] = \mathbb{E}[N_t - N_s] + N_s - \lambda t\\ &= \lambda(t-s) + N_s - \lambda t = M_s.\end{aligned}$$

Problem 6. By definition of the Itô integral, Example 2.5, and the discussion on the quadratic variation (p.16 in the lecture notes),

$$\begin{aligned}Q_\varepsilon(\pi) &= \sum_{i=0}^{k} W_{t_i}(W_{t_{i+1}} - W_{t_i}) + \varepsilon\sum_{i=0}^{k}(W_{t_{i+1}} - W_{t_i})^2\\ &\longrightarrow \int_0^t W_t dW_t + \varepsilon t = W_t^2 + (\varepsilon - 1/2)t\end{aligned}$$

in L^2.

Problem 7. For $t < 1$,

$$\mathbb{E}[W_1|\mathcal{G}_t] = W_1 \neq W_t.$$

Thus $\{W_t\}$ cannot be a martingale w.r.t. $\{\mathcal{G}_t\}_{0\leq t\leq 1}$.

Now, let us compute $\mathbb{E}[W_t - W_s|\mathcal{G}_s]$ explicitly for $0 \leq s < t < 1$. To this end, find that

$$B_t := t(W_{1-1/t} - W_1), \quad t > 0, \quad B_0 := 0$$

is a BM and that $\mathcal{G}_s = \sigma(B_r : 1 \leq r \leq 1/(1-s))$. These facts yield

$$\begin{aligned}\mathbb{E}[W_t - W_s|\mathcal{G}_s] &= \mathbb{E}[(1-t)B_{1/(1-t)} + W_1 - (1-s)B_{1/(1-s)} - W_1|\mathcal{G}_s]\\ &= (1-t)B_{1/(1-s)} - (1-s)B_{1/(1-s)}\\ &= \frac{t-s}{1-s}(W_1 - W_s).\end{aligned}$$

Problem 8. Put $Y_t = aW_t + (ab - a^2/2)t$ to get

$$X_t = e^{Y_t}\int_0^t e^{-Y_s}ds.$$

By the product Itô formula,

$$dX_t = e^{Y_t}e^{-Y_t}dt + \left(\int_0^t e^{-Y_s}ds\right)de^{Y_t}.$$

Further,

$$\begin{aligned} de^{Y_t} &= e^{Y_t} dY_t + \frac{1}{2} e^{Y_t} dY_t dY_t \\ &= e^{Y_t}(adW_t + (ab - a^2/2)dt) + \frac{1}{2} e^{Y_t} a^2 dt = e^{Y_t}(adW_t + abdt). \end{aligned}$$

Thus,

$$dX_t = dt + X_t(adW_t + abdt).$$

Problem 9.

$$dX_t = \begin{pmatrix} 1/2 & 0 \\ 1 & 1/2 \end{pmatrix} X_t dt + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} X_t dW_t$$

Problem 10. First, observe

$$\partial_t u + \frac{1}{2} \partial_{xx}^2 u = 0$$

(see p.19 in the lecture notes). Hence, Itô formula implies

$$u(t, W_t) = 1 + \int_0^t \phi_s dW_s, \quad 0 \le t < 1.$$

Now suppose that

$$\mathbb{E} \int_0^1 \phi_t^2 dt < \infty.$$

Then, by the Itô isometry,

$$\mathbb{E} \int_0^t \phi_s^2 ds = \mathbb{E}[(u(t, W_t) - 1)^2], \quad 0 \le t < 1,$$

whence

$$\mathbb{E} u(t, W_t)^2 \le \mathbb{E} \int_0^1 \phi_s^2 ds + 1, \quad 0 \le t < 1.$$

On the other hand,

$$\mathbb{E} u(t, W_t)^2 = \frac{1}{1-t} \mathbb{E} e^{-W_t^2/(1-t)} \ge \frac{1}{1-t} e^{-t} \mathbb{P}(|W_t| \le \sqrt{t(1-t)}) = \frac{2e^{-t}}{1-t} \int_0^{\sqrt{1-t}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy.$$

Since

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = \frac{d}{dx} \int_0^x \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \Big|_{x=0} = \frac{1}{\sqrt{2\pi}},$$

we deduce

$$\liminf_{t \nearrow 1} \mathbb{E} u(t, W_t)^2 \ge \lim_{t \nearrow 1} \frac{2e^{-t}}{1-t} \int_0^{\sqrt{1-t}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = +\infty,$$

which is a contradiction.