2018 2Q

Wireless Communication Engineering

# #11 Inter Symbol Interference and Adaptive Equalizer

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# Course Schedule (2)

| | Date | Text | Contents |
|---|---|---|---|
| #9 | July 9 | 4.6 | Error correction coding |
| | June 12 | | No class |
| #10 | July 19 | | Adaptive modulation coding |
| #11 | July 23 | 4.3 | Inter symbol interference and adaptive equalizer |
| #12 | July 26 | 3.5 | Orthogonal frequency division multiplexing (OFDM) |
| #13 | July 30 | | Array signal processing and MIMO communications |
| #14 | Aug 2 | | Collaborative exercise for better understanding 2 |
| #15 | TBD | All | Final examination |

# From Previous Lecture

- Throughput against modulation order

$$TP(\gamma, M) = \log_2 M \left(1 - p_{\mathrm{eb}}(\gamma)\right)^L$$

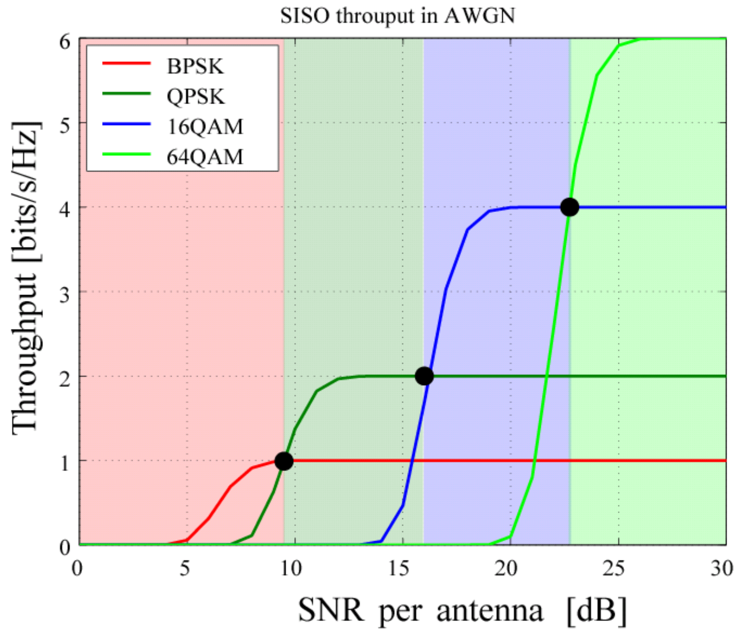
- Adaptive modulation

$$\hat{M} = \arg\max_{M} TP(\gamma, M)$$

- Throughput performance of AMC

$$\overline{TP}(\bar{\gamma}) = \int_0^{\gamma_1} f(\gamma) TP(\gamma, 2) \mathrm{d}\gamma + \cdots + \int_{\gamma_3}^{\infty} f(\gamma) TP(\gamma, 64) \mathrm{d}\gamma$$

SNR Table for AMC

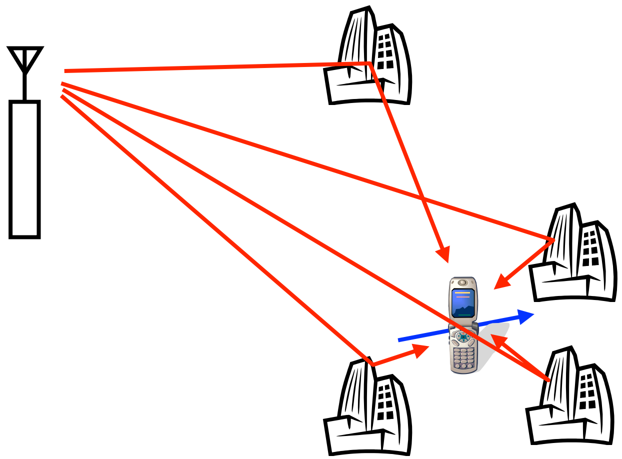


# Contents

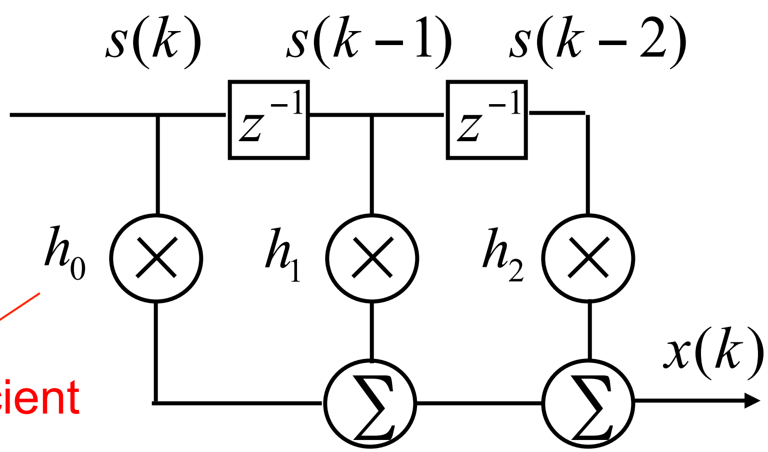
- Delay spread & inter symbol interference
- Classification of equalizer
  - Time domain equalizer (ZF)
  - Frequency domain equalizer (FDE)
- Demonstration

# Multi-path Channel with Delay Spread

Multi-path channel with delay spread



Convolution with channel coefficients




Receive signal model

$$x(t) = \int h(\tau) s(t-\tau) \mathrm{d}\tau + n(t)$$

$$x(k) = \sum_{i=0}^{\infty} h_i s(k-i) + n(k)$$

Convolution between transmit signal & channel response

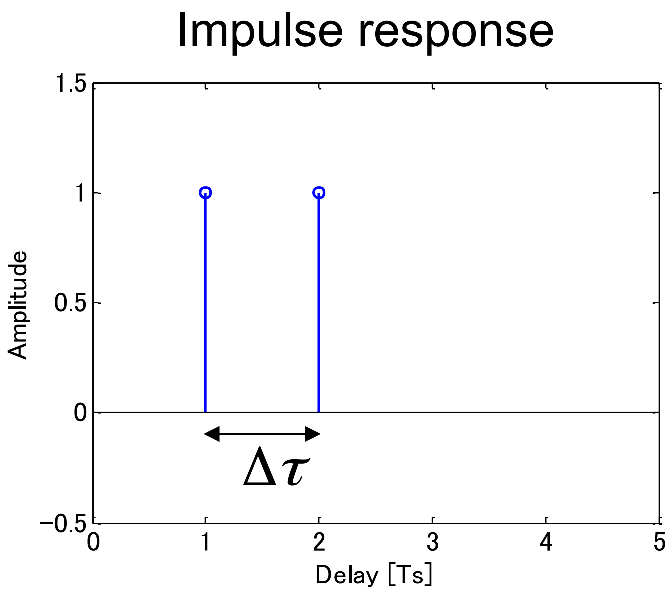
# Delay Spread & Frequency Response

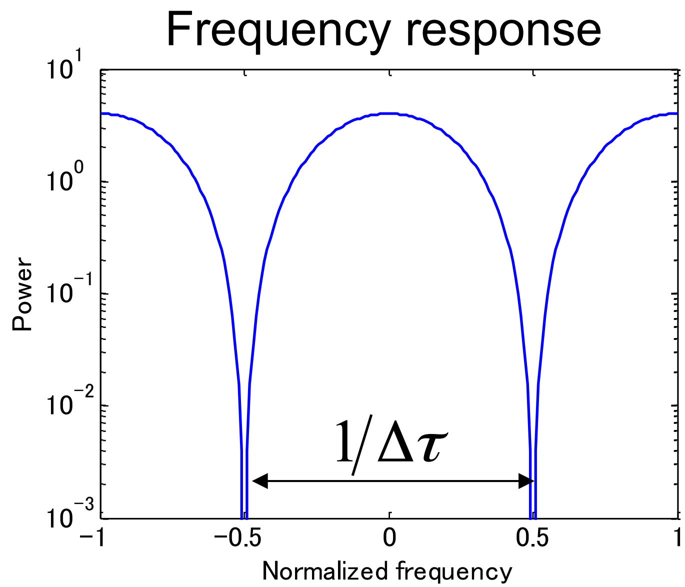
Impulse response

$$h(\tau) = h_0\delta(\tau) + h_{\Delta\tau}\delta(\tau - \Delta\tau)$$

Frequency response

$$H(f) = h_0 + h_{\Delta\tau}\exp(-j2\pi f\Delta\tau)$$



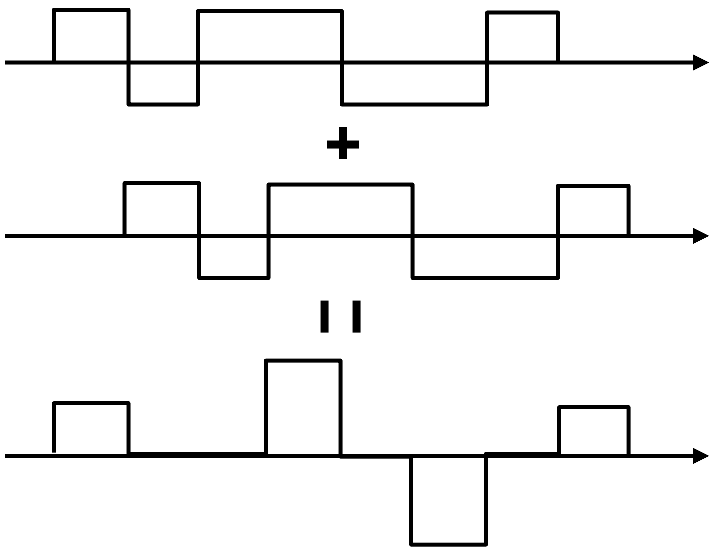

$B > {}^{1}\!/_{\Delta\tau}$ : Wide-band

$B << {}^{1}\!/_{\Delta\tau}$ : Narrow-band

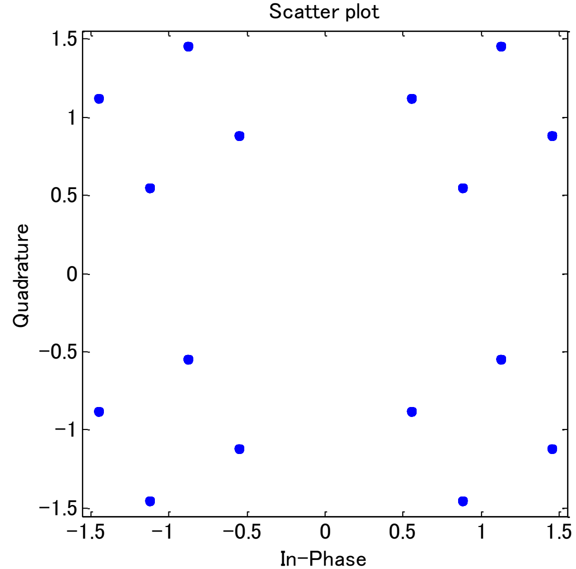
# Inter Symbol Interference (ISI)

## 2-path model

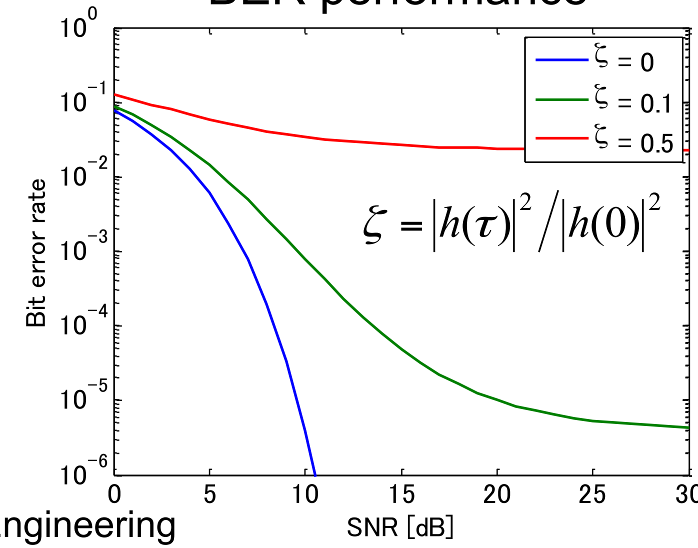
$$x(t) = h(0)s(t) + h(\tau)s(t-\tau) + n(t)$$



## Constellation of QPSK




## BER performance

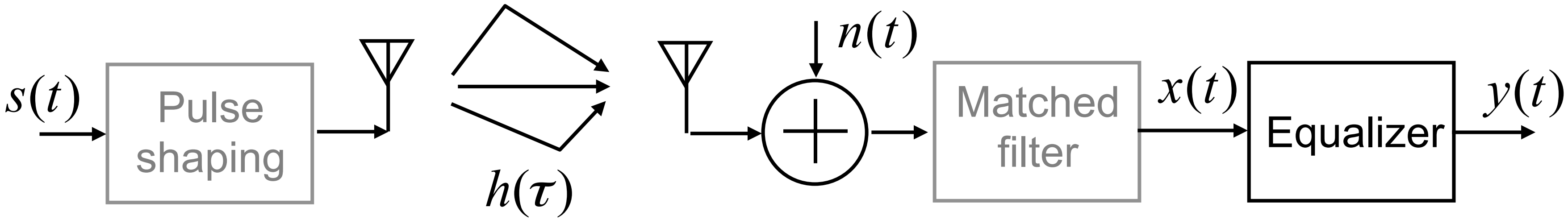



## Signal to Interference plus Noise Ratio (SINR)

$$\gamma = \frac{|h(0)|^2 P}{|h(\tau)|^2 P + \sigma^2}$$

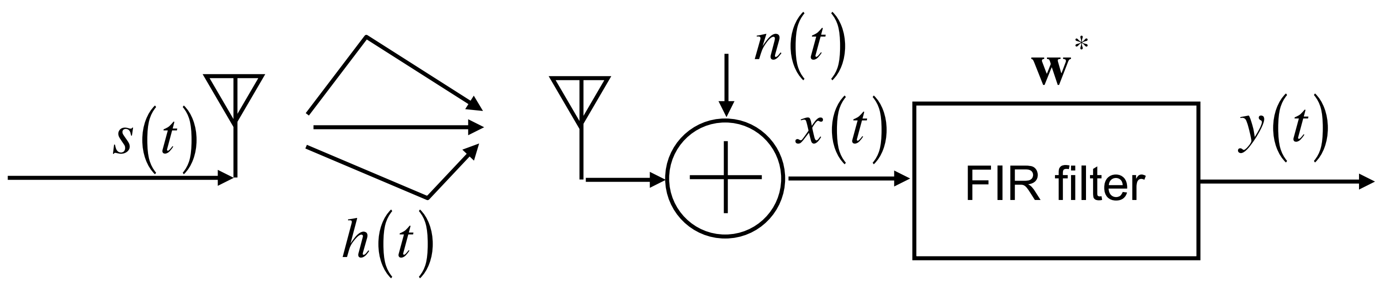
Interference signal power

# Classification of Equalizer




| | Algorithm | Main features |
|---|---|---|
| Linear | Zero Forcing (ZF)<br>Minimum Mean Square Error (MMSE)<br>Frequency Domain Equalizer (FDE) | Inverse frequency response<br>Iterative algorithm (LMS)<br>Frame transmission |
| Nonlinear | Decision Feedback Equalizer (DFE)<br>Maximum Likelihood Sequence Estimation (MLSE) | Infinite Impulse Response (IIR)<br>Viterbi algorithm |

# Time Domain Equalizer




Time domain

$$x(k) = \sum_{i=0}^{\infty} h_i s(k-i) + n(k)$$

$$y(k) = \sum_{i=-\infty}^{\infty} w_i^* x(k-i)$$

$$= w^* \otimes h \otimes s + w^* \otimes n$$

Frequency (Z) domain

$$Y(z) = \sum_{k=-\infty}^{\infty} y(k) z^{-k}, \quad z = e^{j2\pi f}$$

$$Y(z) = W^*(z)H(z)S(z) + W^*(z)N(z)$$

$$W^*(z) = 1/H(z)$$

Channel inversion

$$W^*(z) = \frac{1}{h_0 + h_1 z^{-1} + h_2 z^{-2} + \cdots}$$

Channel inversion requires IIR filter

# Transversal Filter (FIR Filter)

## Transversal (FIR) filter

$$y(k) = \sum_{i=-N}^{N} w_i^* x(k-i)$$

## Z transformation

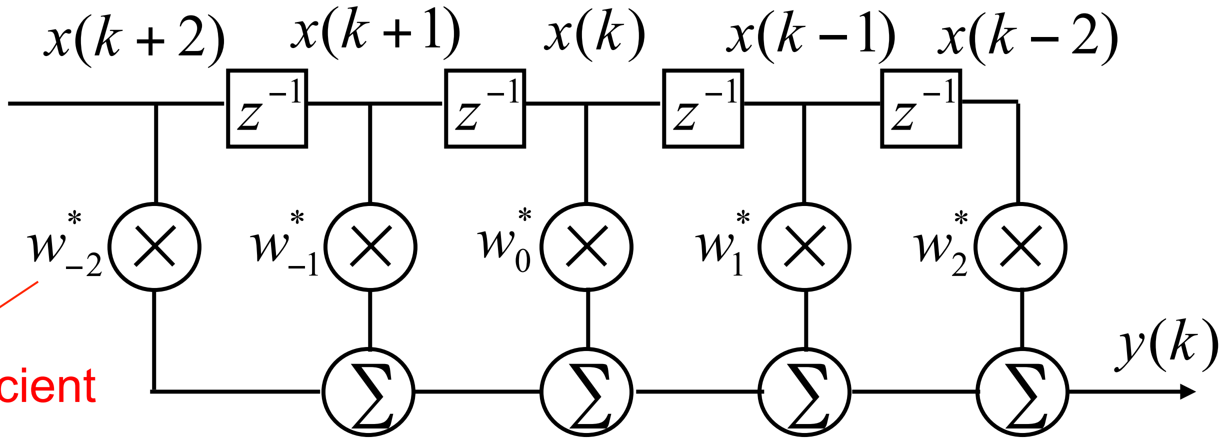
$$W^*(z) = w_{-2}^* + w_{-1}^* z^{-1} + \cdots$$

## Convolution Matrix

$$\begin{bmatrix} y(-2) \\ y(-1) \\ y(0) \\ y(1) \\ y(2) \end{bmatrix} = \begin{bmatrix} x(0) & x(-1) & x(-2) & x(-3) & x(-4) \\ x(1) & x(0) & x(-1) & x(-2) & x(-3) \\ x(2) & x(1) & x(0) & x(-1) & x(-2) \\ x(3) & x(2) & x(1) & x(0) & x(-1) \\ x(4) & x(3) & x(2) & x(1) & x(0) \end{bmatrix} \begin{bmatrix} w_{-2}^* \\ w_{-1}^* \\ w_0^* \\ w_1^* \\ w_2^* \end{bmatrix}$$

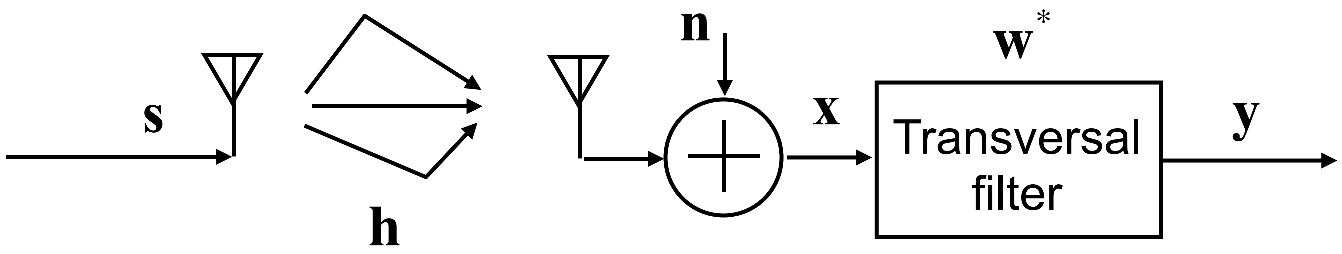
$$\mathbf{y} = \overleftrightarrow{\mathbf{X}} \mathbf{w}^*$$

Cyclic shift matrix




Tap coefficient

# Linear FIR Equalizer (ZF)




**Time domain**

$$x = h \otimes s + n$$

$$y = w^* \otimes h \otimes s + w^* \otimes n$$

**Matrix domain**

$$\mathbf{x} = \overleftrightarrow{\mathbf{S}}\mathbf{h} + \mathbf{n}$$

$$\mathbf{y} = \overleftrightarrow{\mathbf{S}}\overleftrightarrow{\mathbf{H}}\mathbf{w}^* + \overleftrightarrow{\mathbf{N}}\mathbf{w}^*$$

**ZF linear equalizer**

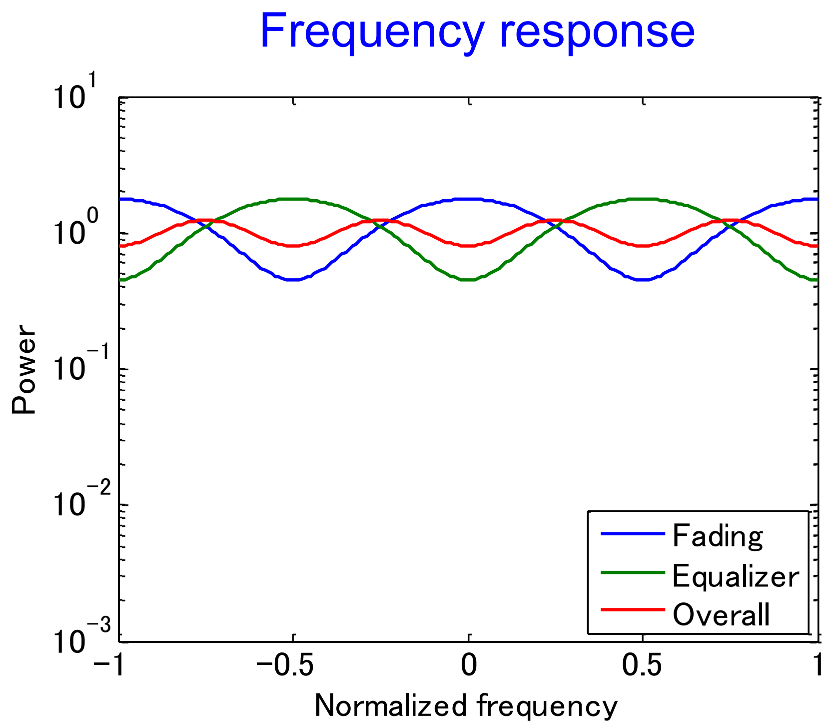
$$\mathbf{w}^* = \overleftrightarrow{\mathbf{H}}^{-1}\mathbf{e}_0$$

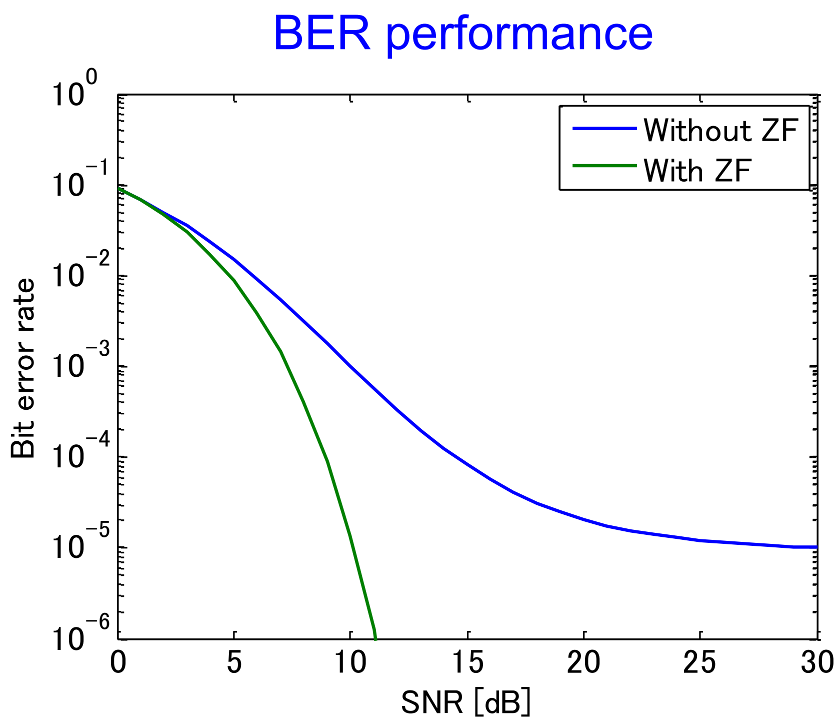
$$\begin{bmatrix} w_{-2}^* \\ w_{-1}^* \\ w_0^* \\ w_1^* \\ w_2^* \end{bmatrix} = \begin{bmatrix} h_o & 0 & 0 & 0 & 0 \\ h_1 & h_o & 0 & 0 & 0 \\ h_2 & h_1 & h_o & 0 & 0 \\ h_3 & h_2 & h_1 & h_o & 0 \\ h_4 & h_3 & h_2 & h_1 & h_o \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Forcing zeros

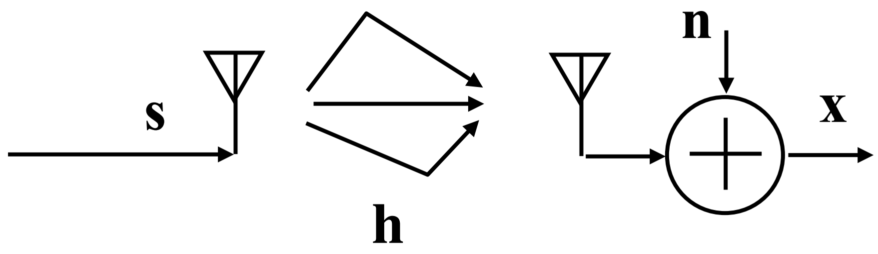
Desired output

# Performance of ZF Equalizer



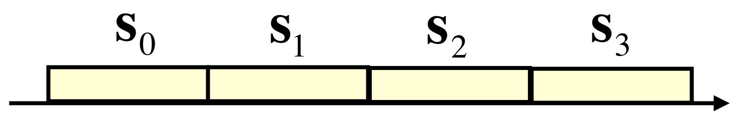



# Frequency Domain Block Signal




Block transmission

$$\mathbf{s} = \begin{bmatrix} s_0 & s_1 & \cdots & s_{L-1} \end{bmatrix}^T$$




Frequency domain transmit signal

$$\tilde{\mathbf{s}} = \mathbf{F}\mathbf{s}, \qquad \tilde{\mathbf{s}} = \begin{bmatrix} \tilde{s}_0 & \tilde{s}_1 & \cdots & \tilde{s}_{L-1} \end{bmatrix}^T$$

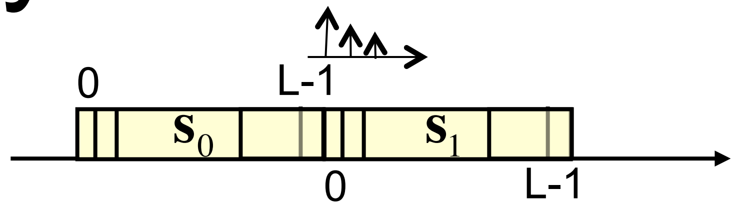
$$\mathbf{F}_{kl} = \frac{1}{\sqrt{L}} \exp\left( -j \frac{2\pi}{L} kl \right)$$

$\mathbf{F}$ : DFT matrix

$\mathbf{F}^{-1} = \mathbf{F}^H$ : IDFT matrix

# Convolution & Cyclic Prefix

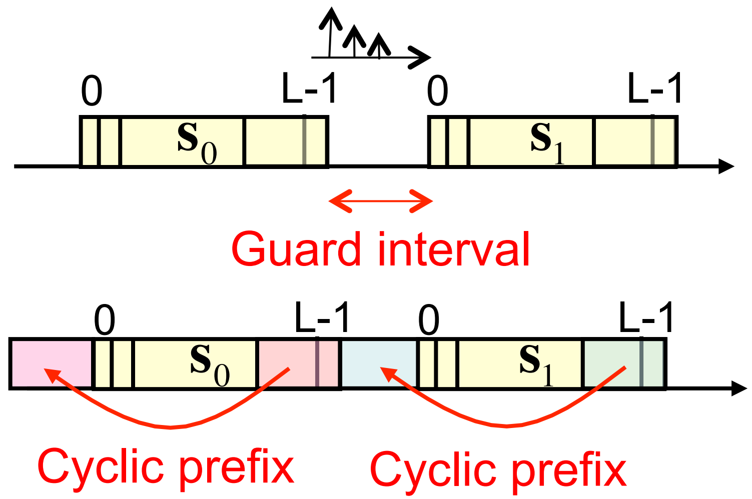
## Without cyclic prefix



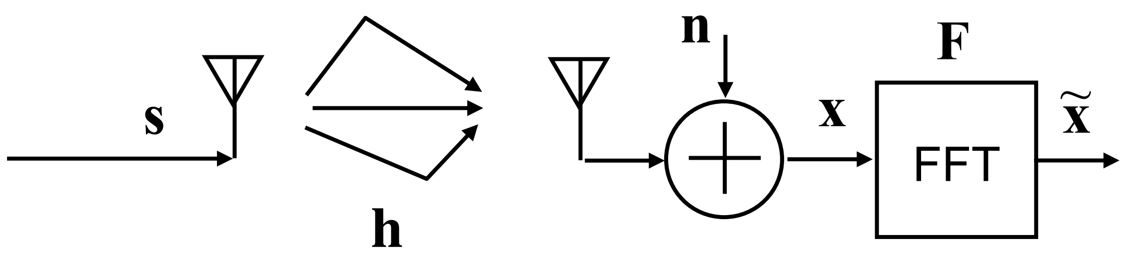

$$\begin{bmatrix} \\ \mathbf{x} \\ \\ \end{bmatrix} = \underbrace{\begin{bmatrix} h_o & 0 & \cdots & \cdots & 0 \\ h_1 & h_o & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ h_{M-1} & \cdots & h_1 & h_o & 0 \\ 0 & h_{M-1} & \cdots & h_1 & h_o \end{bmatrix}}_{\overleftrightarrow{\mathbf{H}}_{11}} \begin{bmatrix} \\ \mathbf{s}_1 \\ \\ \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & h_{M-1} & \cdots & h_1 \\ & & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & h_{M-1} \\ & & & & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}}_{\overleftrightarrow{\mathbf{H}}_{10}} \begin{bmatrix} \\ \mathbf{s}_0 \\ \\ \end{bmatrix}$$

## With cyclic prefix

$$\begin{bmatrix} \\ \mathbf{x} \\ \\ \end{bmatrix} = \underbrace{\begin{bmatrix} h_o & 0 & h_{M-1} & \vdots & h_1 \\ h_1 & h_o & 0 & h_{M-1} & \vdots \\ \vdots & \vdots & \ddots & \vdots & h_{M-1} \\ h_{M-1} & \cdots & h_1 & h_o & 0 \\ 0 & h_{M-1} & \cdots & h_1 & h_o \end{bmatrix}}_{\overleftrightarrow{\mathbf{H}}_{cp}} \begin{bmatrix} \\ \mathbf{s} \\ \\ \end{bmatrix}$$



# Frequency Domain Convolution




Time domain receive signal

$$\mathbf{x} = \overleftrightarrow{\mathbf{H}}_{cp}\mathbf{s} + \mathbf{n}$$

Frequency domain receive signal

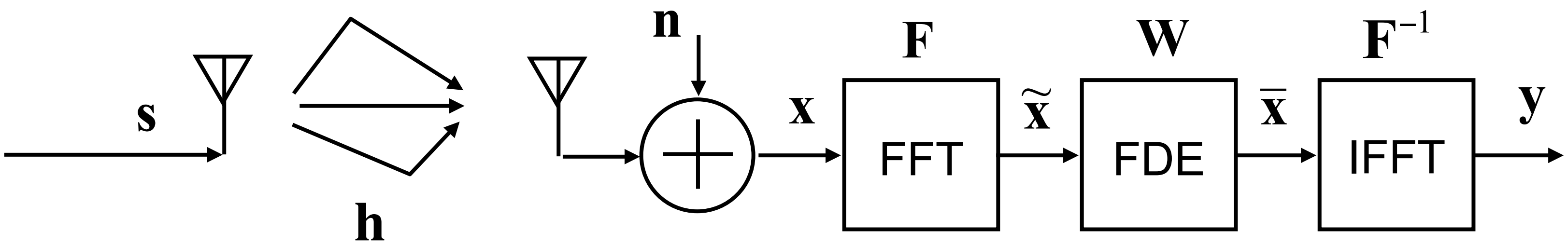
$$\tilde{\mathbf{x}} = \mathbf{F}\mathbf{x} = \mathbf{F}\overleftrightarrow{\mathbf{H}}_{cp}\mathbf{s} + \mathbf{F}\mathbf{n}$$

Frequency domain convolution

Hadamard product

$$\mathbf{F}\overleftrightarrow{\mathbf{H}}_{cp}\mathbf{s} = \mathbf{F}\mathbf{h} \bullet \mathbf{F}\mathbf{s} = \tilde{\mathbf{h}} \bullet \tilde{\mathbf{s}} = \mathrm{diag}\left(\tilde{\mathbf{h}}\right)\tilde{\mathbf{s}} = \begin{bmatrix} \tilde{h}_0 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{h}_{L-1} \end{bmatrix} \begin{bmatrix} \tilde{s}_0 \\ \vdots \\ \tilde{s}_{L-1} \end{bmatrix}$$

# Frequency Domain Equalizer (FDE)




Frequency domain receive signal

$$\tilde{\mathbf{x}} = \mathbf{F}\mathbf{x} = \mathbf{F}\overleftrightarrow{\mathbf{H}}_{\mathrm{cp}}\mathbf{s} + \mathbf{F}\mathbf{n} = \tilde{\mathbf{h}} \bullet \tilde{\mathbf{s}} + \tilde{\mathbf{n}} = \mathrm{diag}\left(\tilde{\mathbf{h}}\right)\tilde{\mathbf{s}} + \tilde{\mathbf{n}}$$

Frequency domain ZF equalizer

$$\mathbf{W} = \begin{bmatrix} \tilde{h}_0 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{h}_{L-1} \end{bmatrix}^{-1} \qquad \tilde{\mathbf{h}} = \mathbf{F}\mathbf{h} = \begin{bmatrix} \tilde{h}_0 & \tilde{h}_1 & \cdots & \tilde{h}_{L-1} \end{bmatrix}^T$$

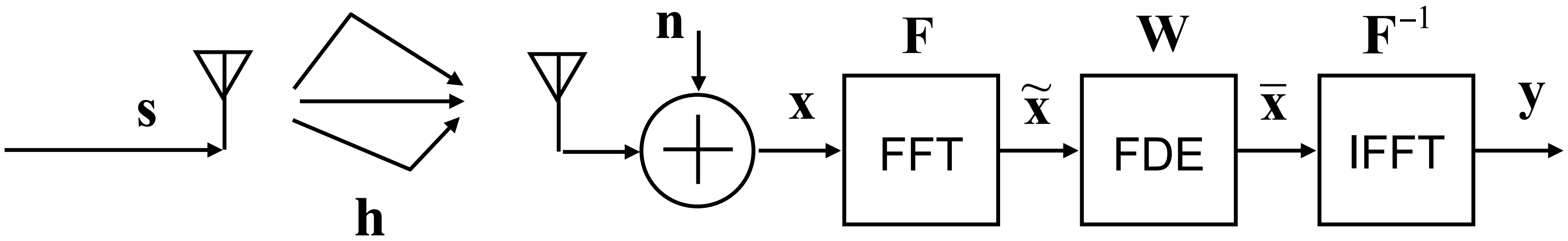
Output of FDE

$$\mathbf{y} = \mathbf{F}^{-1}\mathbf{W}\mathbf{F}\mathbf{x} = \mathbf{F}^{-1}\mathbf{W}\,\mathrm{diag}\left(\tilde{\mathbf{h}}\right)\tilde{\mathbf{s}} + \mathbf{F}^{-1}\mathbf{W}\tilde{\mathbf{n}} = \mathbf{F}^{-1}\tilde{\mathbf{s}} + \mathbf{n}' = \mathbf{s} + \mathbf{n}'$$

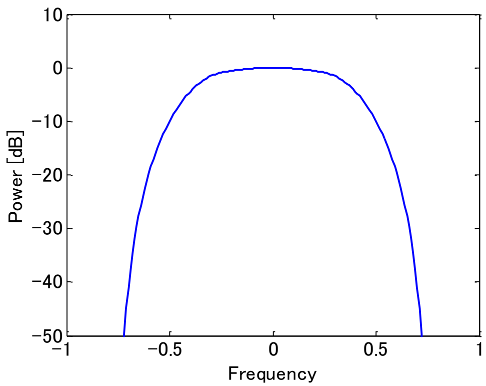
ISI free

Colored noise

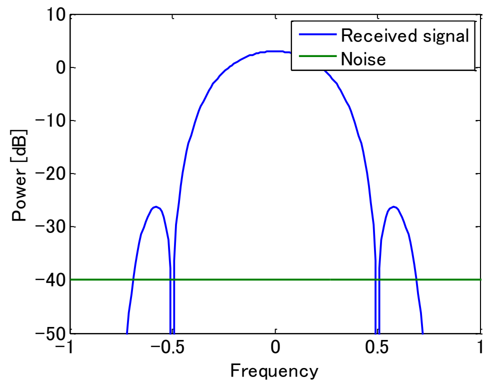
# Frequency Domain Equalizer (FDE)



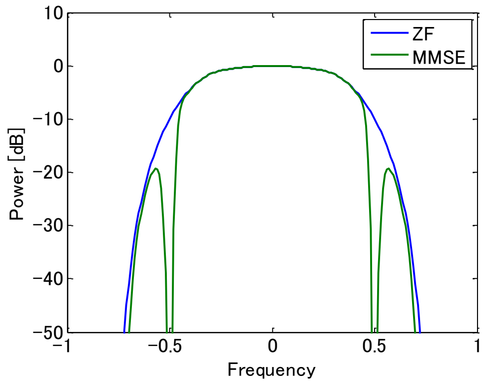

## Transmit spectrum



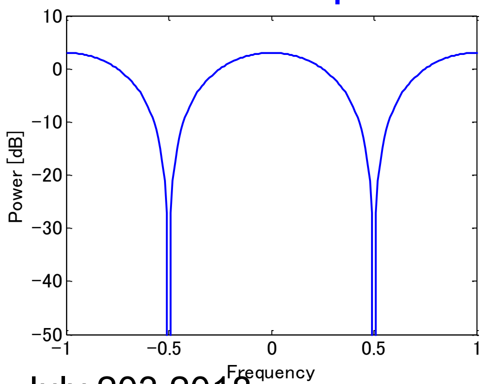

## Receive spectrum



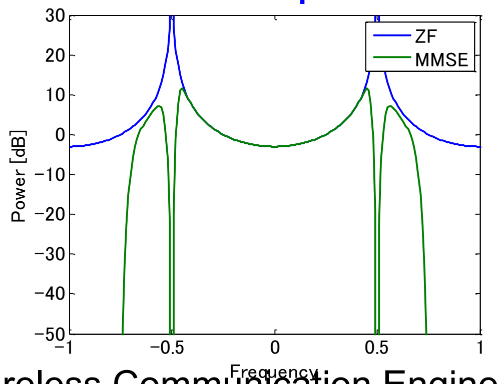

## Equalized spectrum



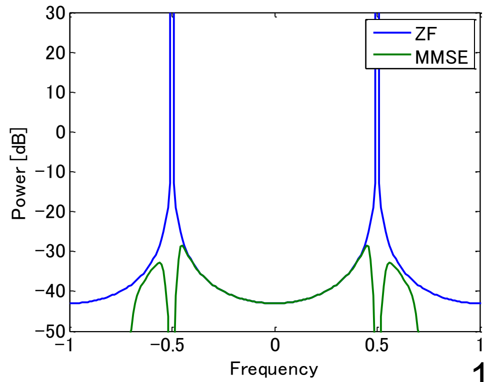

## Channel response




## FDE response



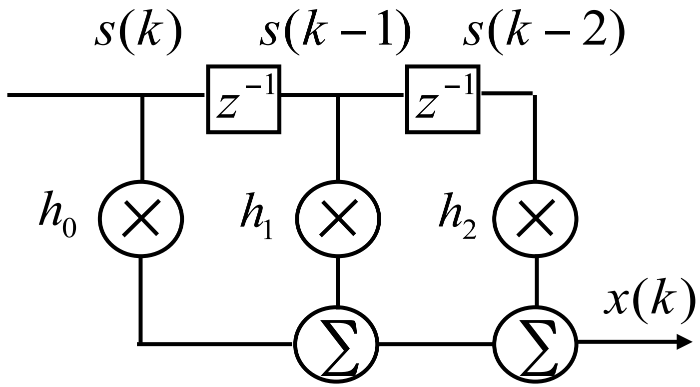

## Noise spectrum (ZF, MMSE)



# Summary

- Multi-path channel with delay spread

$$x(k) = \sum_{i=0}^{\infty} h_i s(k-i) + n(k)$$




- Time domain equalizer（ZF）

$$y(k) = \sum_{i=-\infty}^{\infty} w_i^* x(k-i)$$

$$\mathbf{y} = \overleftrightarrow{\mathbf{S}}\overleftrightarrow{\mathbf{H}}\mathbf{w}^* + \overleftrightarrow{\mathbf{N}}\mathbf{w}^* \longrightarrow \mathbf{w}^* = \overleftrightarrow{\mathbf{H}}^{-1}\mathbf{e}_0$$

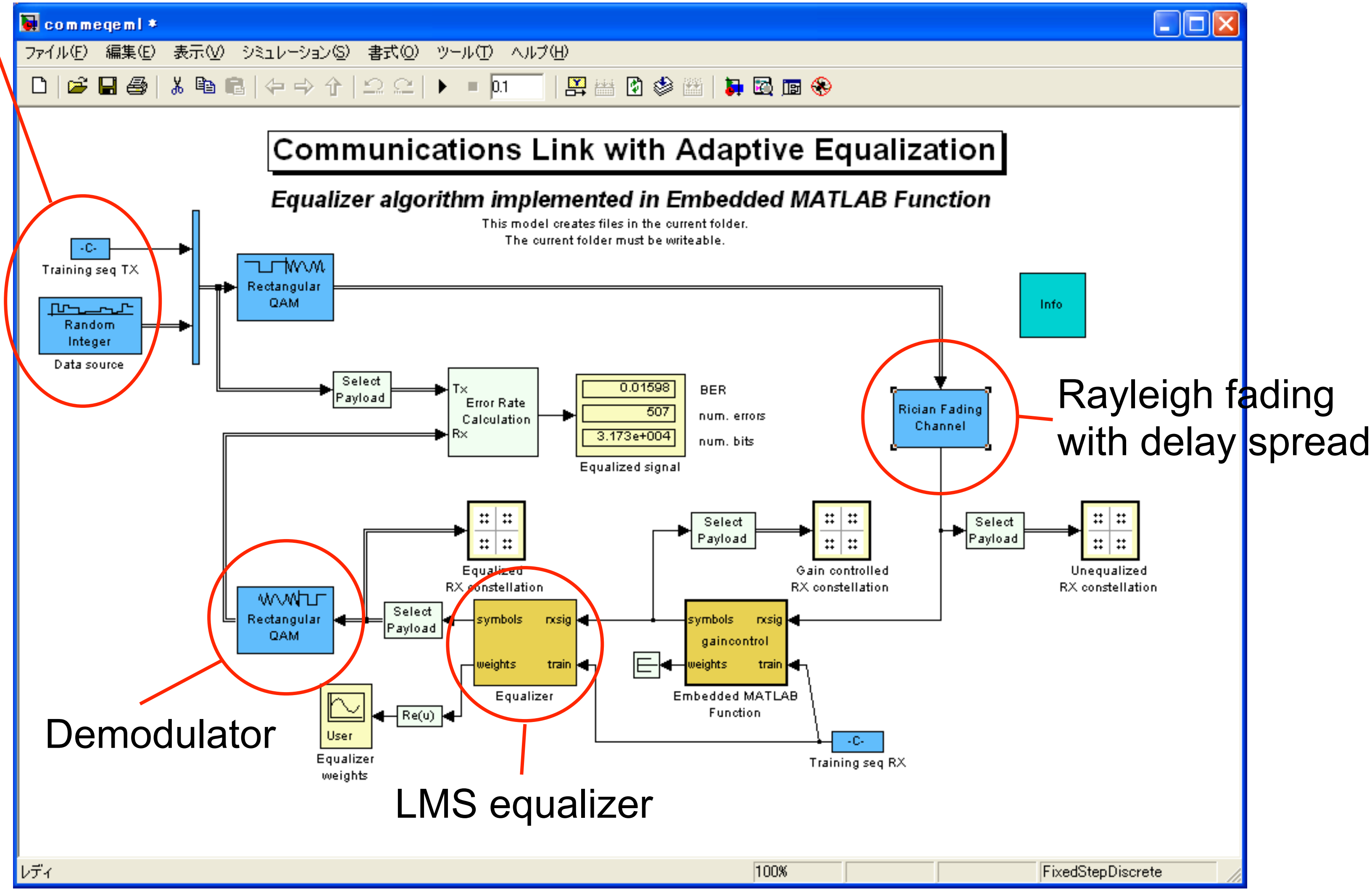
- Frequency domain equalizer（FDE）

$$\mathbf{y} = \mathbf{F}^{-1}\mathbf{W}\mathbf{F}\mathbf{x}$$

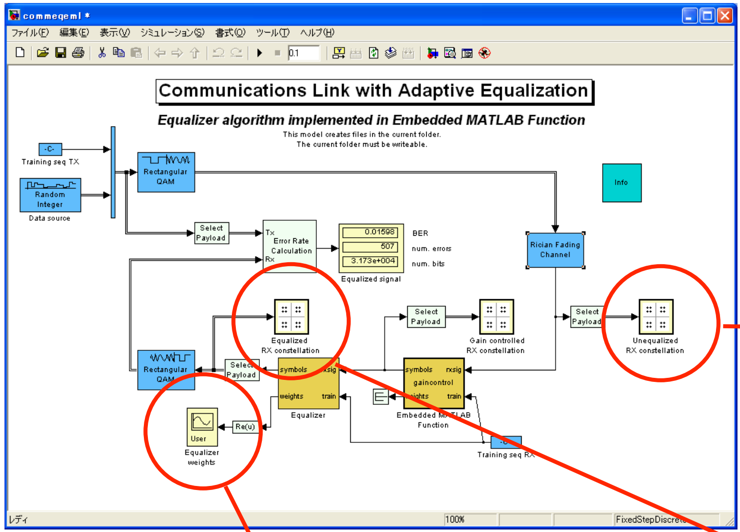
$$\mathbf{F}\mathbf{x} = \mathbf{F}\overleftrightarrow{\mathbf{H}}_{cp}\mathbf{s} = \tilde{\mathbf{h}} \bullet \tilde{\mathbf{s}} \longrightarrow \mathbf{W} = \mathrm{diag}\begin{bmatrix} 1/\tilde{h}_0 & 1/\tilde{h}_1 & \cdots & 1/\tilde{h}_{N-1} \end{bmatrix}$$

# Demo

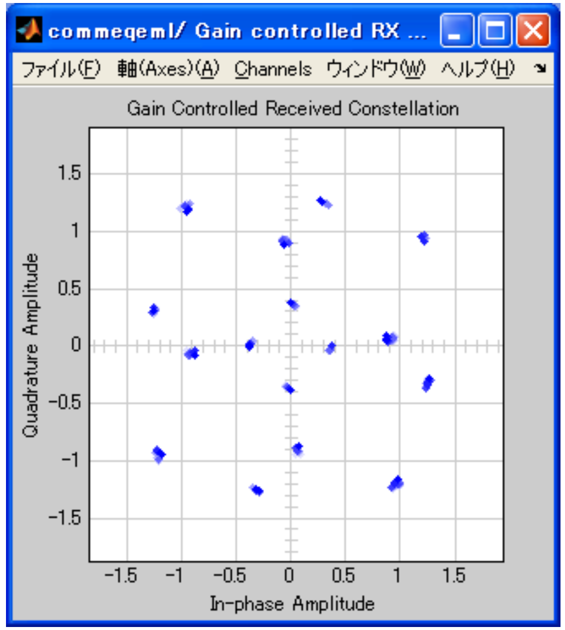
Modulator＋Training sequence



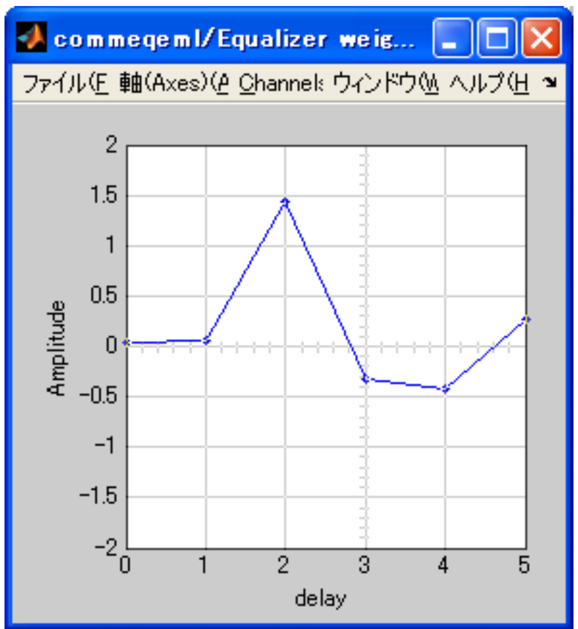
# Demo



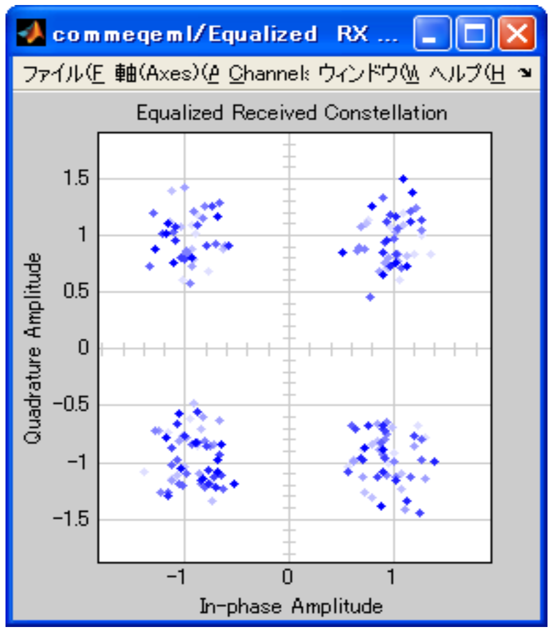

Before equalizer




Tap coefficients

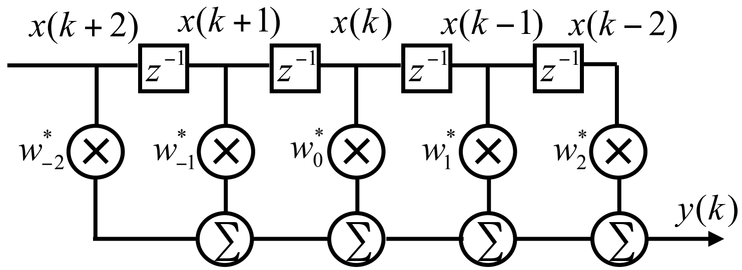



After equalizer



# Linear Equalizer (MMSE)

Mean Square Error (MSE) of symbol




$$\hat{s}(k) = \sum_{i=-\infty}^{\infty} w_i^* x(k-i) = \mathbf{w}^H \mathbf{x}$$

$$J = \mathrm{E}[|e(k)|^2] = E[|s(k) - \hat{s}(k)|^2]$$

$$= E[\mathbf{w}^H \mathbf{x}\mathbf{x}^H \mathbf{w} - 2\,\mathrm{Re}[\mathbf{w}^H \mathbf{x} s^*(k)] + |s(k)|^2]$$

$$= \mathbf{w}^H \mathbf{R}_x \mathbf{w} - 2P\,\mathrm{Re}[\mathbf{w}^H \mathbf{h}] + P$$

$$\mathbf{R}_x = \mathrm{E}[\mathbf{x}\mathbf{x}^H] \qquad P\mathbf{h} = \mathrm{E}[\mathbf{x} s^*(k)]$$

Covariance matrix — Channel estimation

MMSE equalizer

$$\frac{\partial}{\partial \mathbf{w}} J = 2\mathbf{R}_x \mathbf{w} - 2P\mathbf{h} = 0$$

$$\mathbf{w} = P\mathbf{R}_x^{-1}\mathbf{h}$$

Winner filter

Derivation of complex matrix

$$\frac{\partial \alpha}{\partial w} = \frac{\partial \alpha}{\partial w_{\mathrm{I}}} + j\frac{\partial \alpha}{\partial w_{\mathrm{Q}}}$$

$$\frac{\partial}{\partial \mathbf{w}} \mathbf{w}^H \mathbf{h} = 2\mathbf{h}$$

$$\frac{\partial}{\partial \mathbf{w}} \mathbf{w}^H \mathbf{R}_x \mathbf{w} = 2\mathbf{R}_x \mathbf{w}$$

# Least Mean Square (LMS)

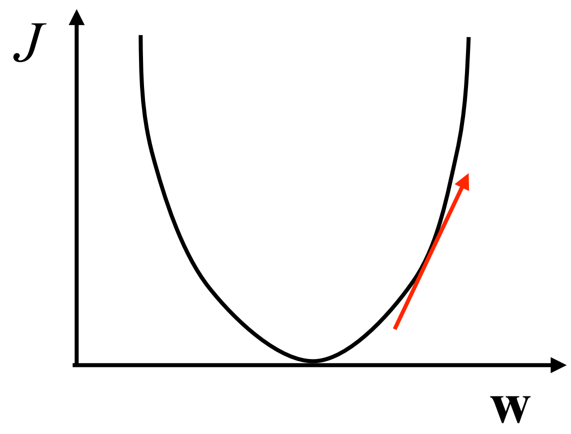
Derivative of MSE

$$\frac{\partial}{\partial \mathbf{w}} J = 2\mathbf{R}_x \mathbf{w} - 2\mathbf{h}$$

$$= 2\,\mathrm{E}\left[\mathbf{x}\left(\mathbf{x}^H \mathbf{w} - s^*(k)\right)\right]$$

$$= -2\,\mathrm{E}\left[\mathbf{x} e^*(k)\right]$$

Iterative optimization




With optimal weight

$$\mathrm{E}\left[\mathbf{x}^* e(k)\right] = 0$$

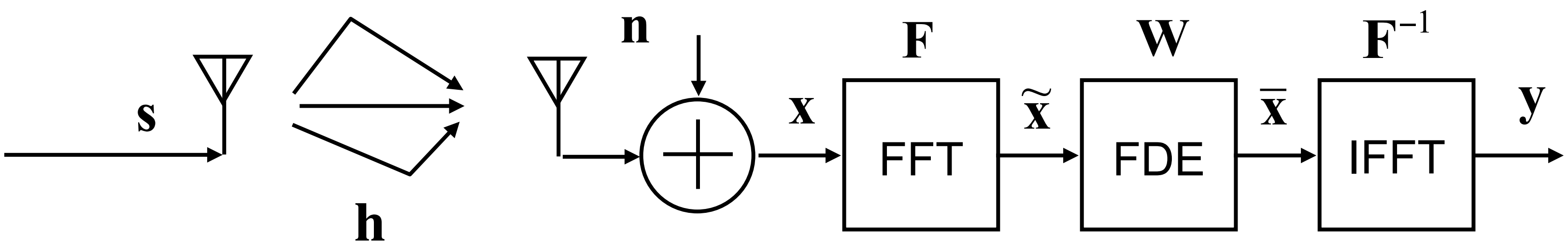
Iterative MMSE Optimization (LMS)

$$\mathbf{w}(k+1) = \mathbf{w}(k) - \mu \frac{\partial J}{\partial \mathbf{w}}$$

$$= \mathbf{w}(k) - \mu \mathbf{x}(k+1) e^*(k)$$

Without calculation of
covariance matrix and channel vector

# FDE (MMSE)




Frequency domain receive signal

$$\tilde{\mathbf{x}} = \mathbf{F}\mathbf{x} = \mathbf{F}\overleftrightarrow{\mathbf{H}}_{\mathrm{cp}}\mathbf{s} + \mathbf{F}\mathbf{n} = \tilde{\mathbf{h}} \bullet \tilde{\mathbf{s}} + \tilde{\mathbf{n}}$$

$$\tilde{\mathbf{h}} = \mathbf{F}\mathbf{h} = \begin{bmatrix} \tilde{h}_0 & \tilde{h}_1 & \cdots & \tilde{h}_{L-1} \end{bmatrix}^T \quad \tilde{\mathbf{s}} = \mathbf{F}\mathbf{s} = \begin{bmatrix} \tilde{s}_0 & \tilde{s}_1 & \cdots & \tilde{s}_{L-1} \end{bmatrix}^T$$

ZF equalizer

$$\mathbf{w} = \begin{bmatrix} 1/\tilde{h}_0 & 1/\tilde{h}_1 & \cdots & 1/\tilde{h}_{L-1} \end{bmatrix}^T$$

MMSE equalizer

$$\mathbf{w} = \begin{bmatrix} \dfrac{\tilde{P}\tilde{h}_0^*}{\tilde{P}\left|\tilde{h}_0\right|^2 + \tilde{\sigma}^2} & \cdots & \dfrac{\tilde{P}\tilde{h}_{L-1}^*}{\tilde{P}\left|\tilde{h}_{L-1}\right|^2 + \tilde{\sigma}^2} \end{bmatrix}^T$$

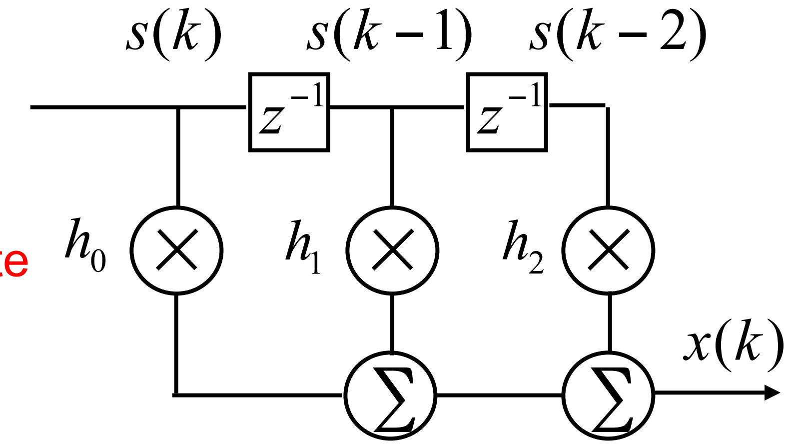
Output of FDE

$$\mathbf{y} = \mathbf{F}^{-1}\mathbf{W}\mathbf{F}\mathbf{x} \qquad \mathbf{W} = \mathrm{diag}\left[\mathbf{w}\right]$$

# Maximum Likelihood Estimation

**Receive signal**

$$x(k) = \sum_{i=0}^{2} h_i s(k-i) + n(k)$$




**Likelihood function**

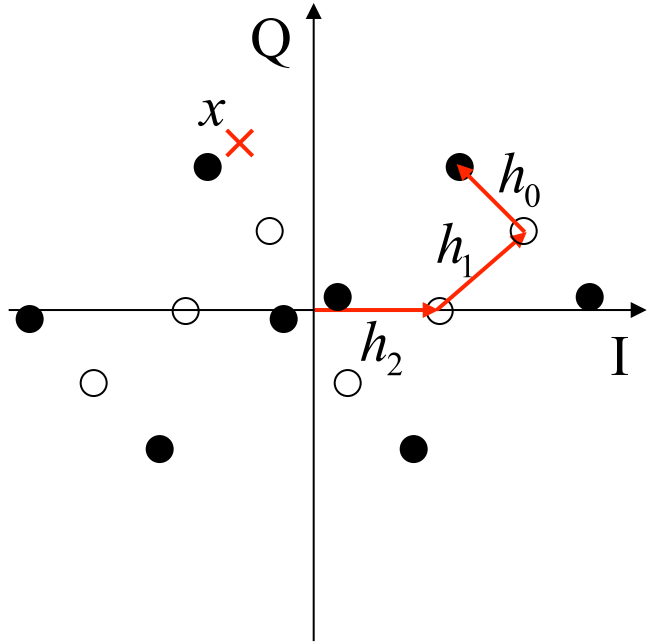
$$J(k) = \left| x(k) - \sum_{i=0}^{2} h_i \tilde{s}(k-i) \right|^2$$

Symbol candidate

Replica signal

**Maximum likelihood estimation**

$$\hat{s}(k), \hat{s}(k-1), \hat{s}(k-2) = \arg \min_{\tilde{s}(k), \tilde{s}(k-1), \tilde{s}(k-2)} J(k)$$




**Complexity**

Modulation order $M$

Number of taps $L$

$M^L$ search problem

# Maximum Likelihood Sequence Estimation (MLSE)

## Receive signal

$$x(k) = \sum_{i=0}^{2} h_i s(k-i) + n(k)$$

## Branch metric

$$B_{\mathrm{AA}}(k) = x(k) - h_0 - h_1 - h_2$$

$$B_{\mathrm{AC}}(k) = x(k) - h_0 - h_1 + h_2$$

## Path metric

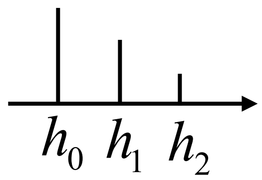
$$P_{\mathrm{AA}}(k) = P_{\mathrm{A}}(k-1) + B_{\mathrm{AA}}(k)$$

$$P_{\mathrm{AC}}(k) = P_{\mathrm{A}}(k-1) + B_{\mathrm{AC}}(k)$$

## Survived path

$$P_{\mathrm{A}}(k) = \min\left[P_{\mathrm{AA}}(k), P_{\mathrm{AC}}(k)\right]$$

3-path model




Trellis diagram in the case of BPSK

$$A = \begin{pmatrix}-1 & -1\end{pmatrix} \quad B = \begin{pmatrix}1 & -1\end{pmatrix}$$

$$C = \begin{pmatrix}-1 & 1\end{pmatrix} \quad D = \begin{pmatrix}1 & 1\end{pmatrix}$$