2018 2Q

Wireless Communication Engineering

#9 Error Correction Coding

Kei Sakaguchi

sakaguchi@mobile.ee.

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Course Schedule (2)

	Date	Text	Contents
#9	July 9	4.6	Error correction coding
	June 12		No class
#10	July 19		Adaptive modulation coding
#11	July 23	4.3	Inter symbol interference and adaptive equalizer
#12	July 26	3.5	Orthogonal frequency division multiplexing (OFDM)
#13	July 30		Array signal processing and MIMO communications
#14	Aug 2		Collaborative exercise for better understanding 2
#15	TBD	All	Final examination

From Previous Lecture

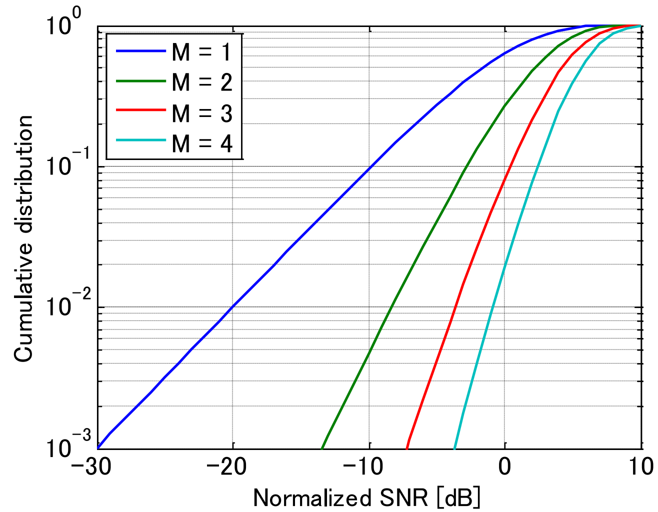
- Error rate in fading channel

$$p_{\rm eb}(\gamma) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\gamma}\right) \quad \Rightarrow \quad \bar{p}_{\rm eb}(\bar{\gamma}) = \frac{1}{2}\left(1 - \sqrt{\frac{\bar{\gamma}}{1+\bar{\gamma}}}\right)$$

- Antenna signal processing

$$\mathbf{y} = \mathbf{h}s + \mathbf{n}$$

$$\hat{s} = \mathbf{w}^H\mathbf{y} = \mathbf{w}^H\mathbf{h}s + \mathbf{w}^H\mathbf{n}$$



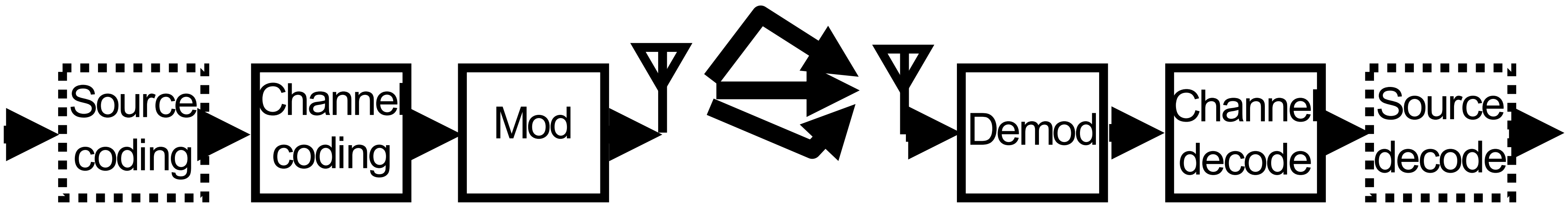
- Performance of MRC diversity

$$\gamma_{\rm opt} = \frac{|\mathbf{h}|^2 P}{\sigma^2} = \frac{\sum_{i=1}^{M} |h_i|^2 P}{\sigma^2} = \sum_{i=1}^{M} \gamma_i \quad \Rightarrow \quad f(\gamma) = \frac{1}{(M-1)!\,\bar{\gamma}^M}\gamma^{M-1}\exp\left(-\frac{\gamma}{\bar{\gamma}}\right)$$

Contents

- Structure of channel coding
- Block coding
- Convolutional coding
- Viterbi decoder (hard decision)
- Viterbi decoder (soft decision)
- Puncture & eraser
- Fading & interleaver

Source Coding & Channel Coding



Source coding

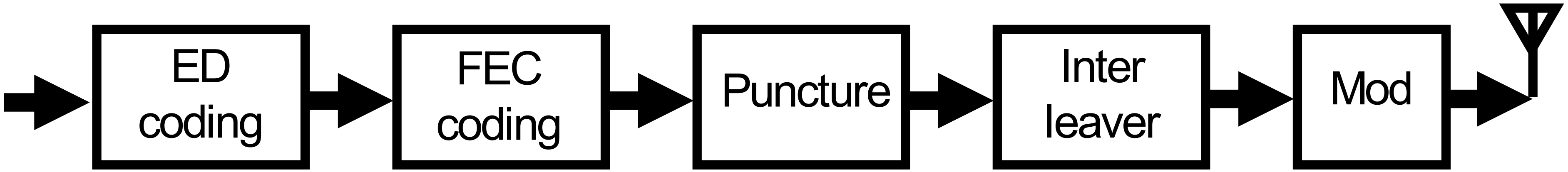
Coding to compress information by removing redundancy

(e.g. zip, mpeg, etc.)

Channel coding

Coding to make it tolerant against error by adding redundancy

Structure of Channel Coding



Error Detection (ED) coding

Coding to detect error event at the receiver, the detected information is used for Automatic Repeat reQuest（ARQ）(e.g. even parity, CRC)

Forward Error Correction (FEC) coding

Coding to make it tolerant against error by adding redundancy (e.g. block coding (Hamming, Read Solomon), convolutional coding (CC, Turbo)

Puncture

To control coding rate by removing parity bits adaptively

Interleaver

Randomizer to avoid burst error due to fading

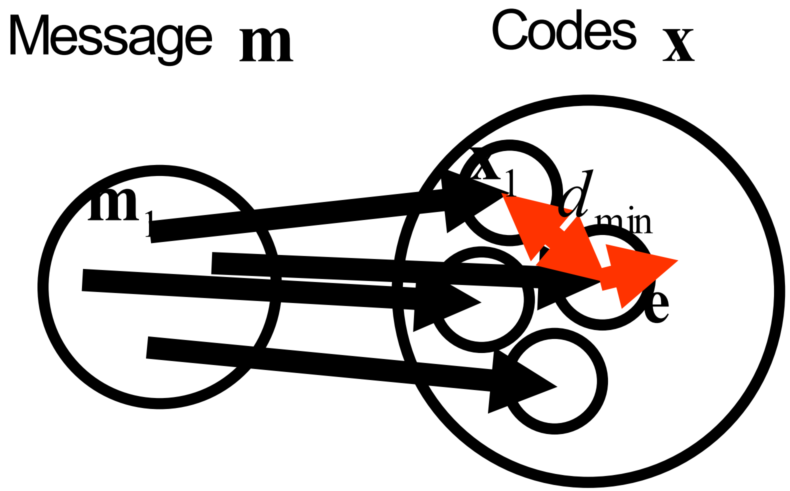
Block Coding

(n, k) block coding

Message sequence $\mathbf{m} = [m_1 \quad m_2 \quad \cdots \quad m_k]$

Coded sequence $\mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_n]$

Coding rate $r = k/n$



Linear coding

$$\mathbf{x}_i \oplus \mathbf{x}_j = \mathbf{x}_k$$

Hamming distance

$$\mathrm{ham}(\mathbf{x}_i, \mathbf{x}_j) = \mathrm{hamw}(\mathbf{x}_i \oplus \mathbf{x}_j) = \mathrm{hamw}(\mathbf{x}_k)$$

Potential of error correction

$$d_{\min} = \min_{i,j} \mathrm{ham}(\mathbf{x}_i, \mathbf{x}_j) = \min_i \mathrm{hamw}(\mathbf{x}_i)$$

$$d_c = \left\lfloor \frac{1}{2}(d_{\min} - 1) \right\rfloor$$

(3, 1) repetition code

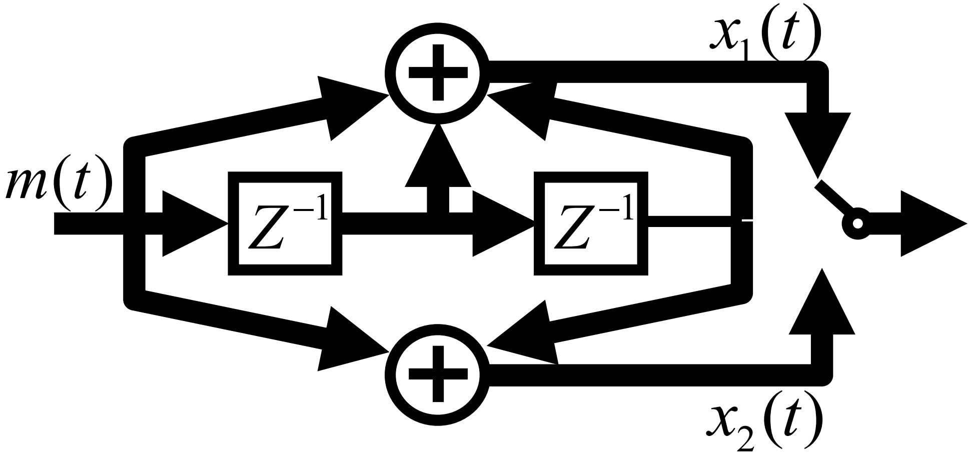
Message		Code
0	→	0 0 0
1	→	1 1 1

$r = 1/3$

Minimum distance $d_{\min} = 3$

Potential of error correction $d_c = \left\lfloor \frac{1}{2}(3-1) \right\rfloor = 1$

Convolutional Coding



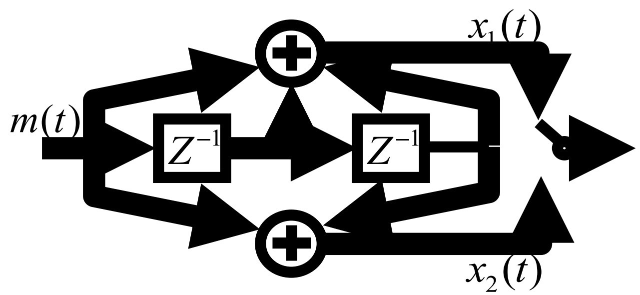
Coding rate $r = 1/2$

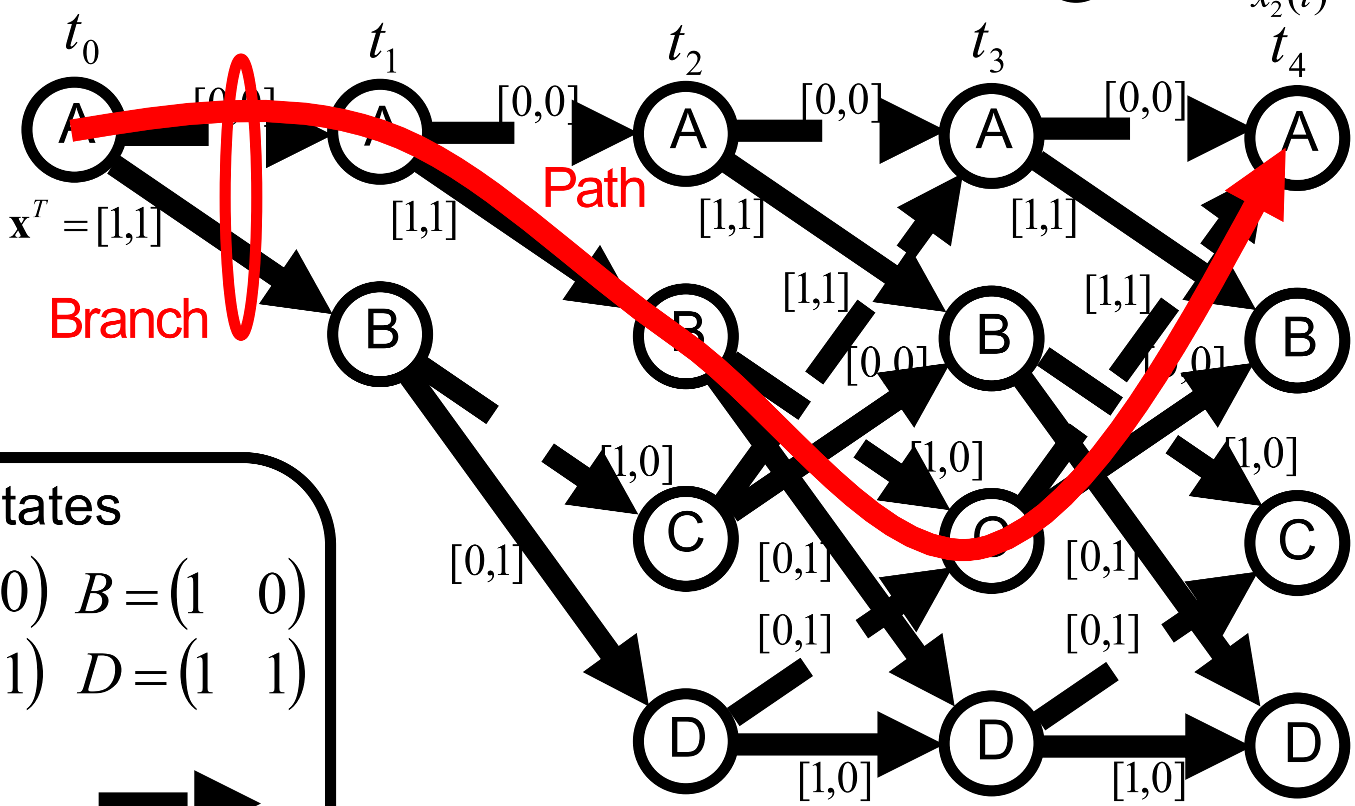
Constraint length $N = 3$

Generator polynomial $g_1 = 7, g_2 = 5$

Input (messages)	Output (codes)
0 0 0 0	00 00 00 00
0 1 0 0	00 11 10 11

Trellis Graph





States

$A = (0 \quad 0)$ $B = (1 \quad 0)$

$C = (0 \quad 1)$ $D = (1 \quad 1)$

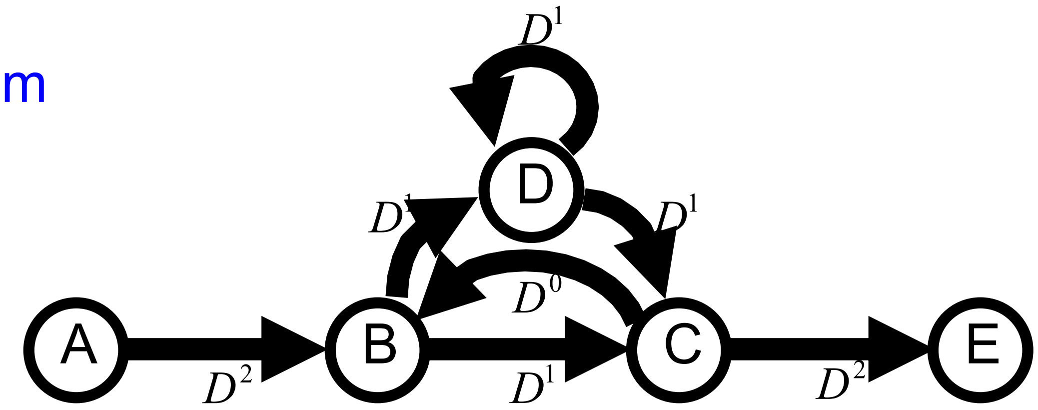
Input 0

Input 1

Input	Output
0 1 0 0	00 11 10 11

State Diagram & Transfer Function

State diagram



Transfer function

$$X_B = D^2 X_A + X_C$$
$$X_C = D X_B + D X_D$$
$$X_D = D X_B + D X_D$$
$$X_E = D^2 X_C$$

$$T(D) = X_E / X_A = \frac{D^5}{1-2D}$$
$$= D^5 + 2D^6 + 4D^7 + \ldots + 2^{d-d_{\min}} D^d$$

Minimum distance $d_{\min} = 5$

Potential error correction $d_c = \left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor$

Viterbi Decoder (Hard Decision)

Receive signal

$$\mathbf{r}(t) = [r_1(t), r_2(t)]^T$$

Branch metric

$$b_{AA}(t) = \mathrm{ham}\left(\mathbf{r}(t), \mathbf{x}_{AA}\right)$$

$$b_{AC}(t) = \mathrm{ham}\left(\mathbf{r}(t), \mathbf{x}_{AC}\right)$$

$$\mathrm{ham}\left(\mathbf{a}, \mathbf{b}\right) = \sum_i a_i \oplus b_i$$

Path metric

$$p_{AA}(t) = p_A(t-1) + b_{AA}(t)$$

$$p_{AC}(t) = p_C(t-1) + b_{AC}(t)$$

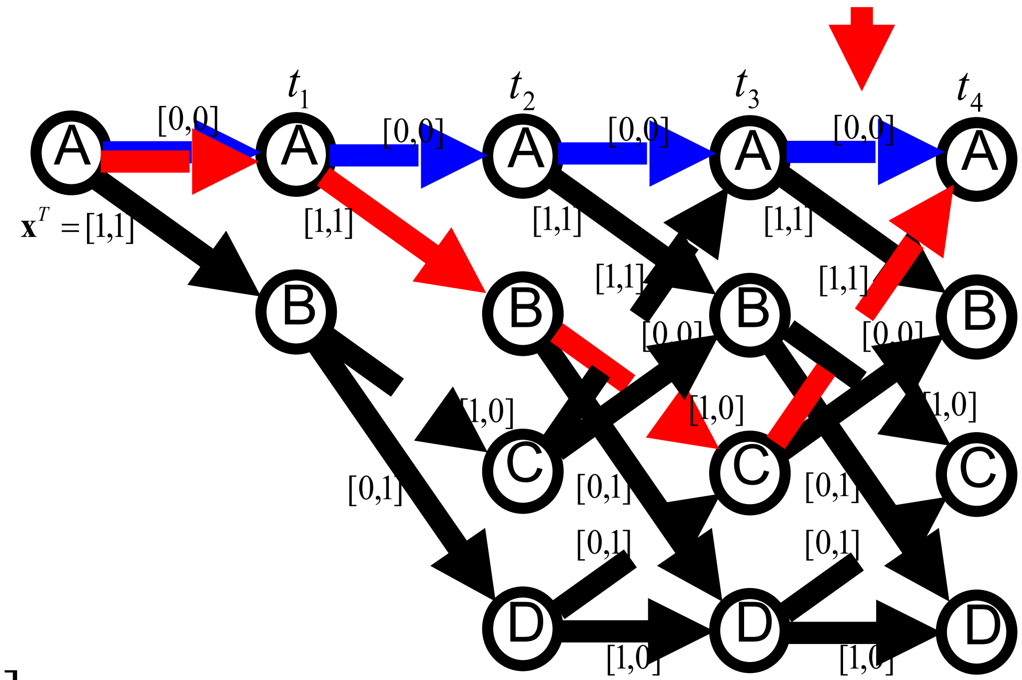
Survived path

$$p_A(t) = \min[p_{AA}(t), p_{AC}(t)]$$

Path decision

$$\hat{m}(t) = \arg\min[p_A(t), p_B(t), p_C(t), p_D(t)]$$

Decoded sequence



Example of Viterbi Decoding

Message sequence

Tail bits

$$\mathbf{m} = [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]$$

Coded sequence

$$\mathbf{x} = [00 \quad 11 \quad 10 \quad 11 \quad 00 \quad 00]$$

Receive signal

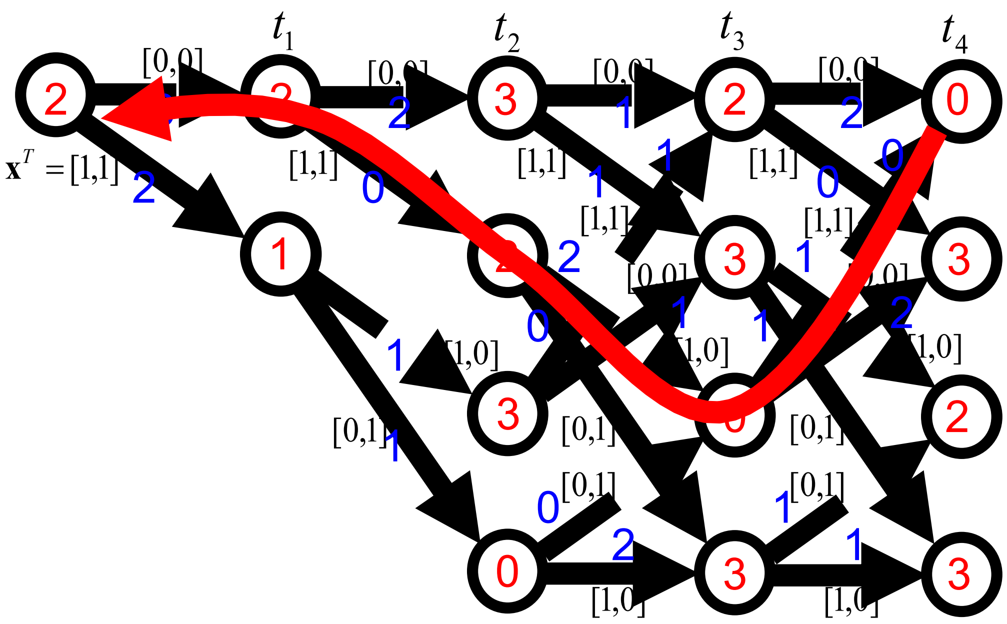
Error event

$$\mathbf{r} = [00 \quad 11 \quad 01 \quad 11 \quad 00 \quad 00]$$

Decoded message

$$\hat{\mathbf{m}} = [0 \quad 1 \quad 0 \quad 0]$$

(Hard decision) Viterbi decoding



Complexity of Viterbi Decoding

Convolutional coding

$$(n,\ k) = (8,\ 4)$$

$$N = 3$$

Total search space

$$L = 2^n = 256$$

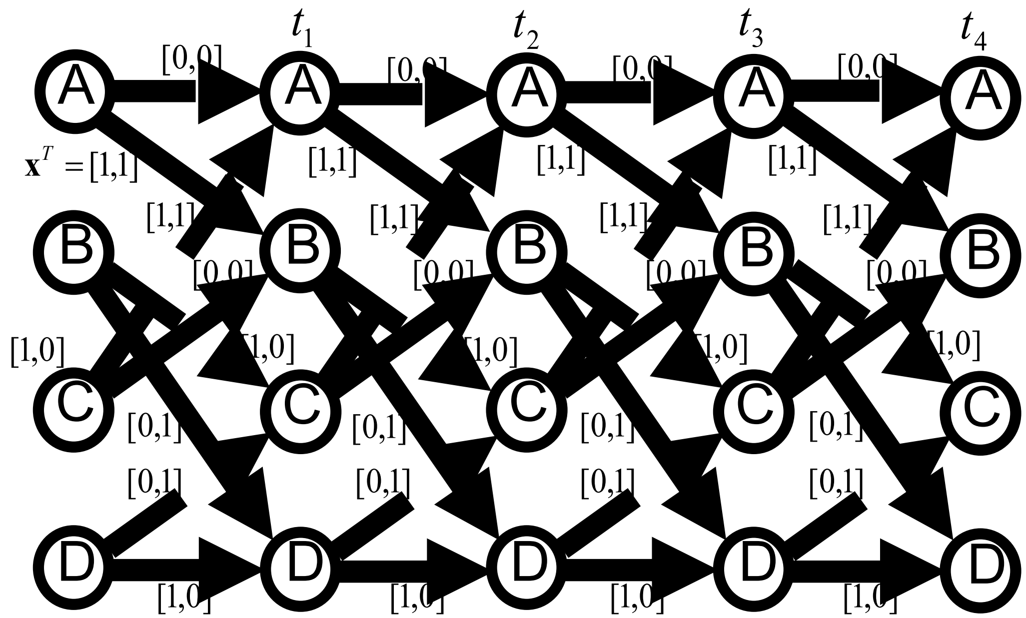
Complexity of Trellis search

$$L = 2^{N-1} \cdot 2^k = 64$$

Complexity of Viterbi decoding

$$L = 2^{N-1} \cdot 2 \cdot k = 32$$

Trellis graph



Error Rate of Viterbi Decoding

Hamming distance between codes

$$d = \mathrm{ham}(\mathbf{x}_1, \mathbf{x}_2)$$

$$\mathbf{x}_1 = [00 \quad 00 \quad 00 \quad 00]$$

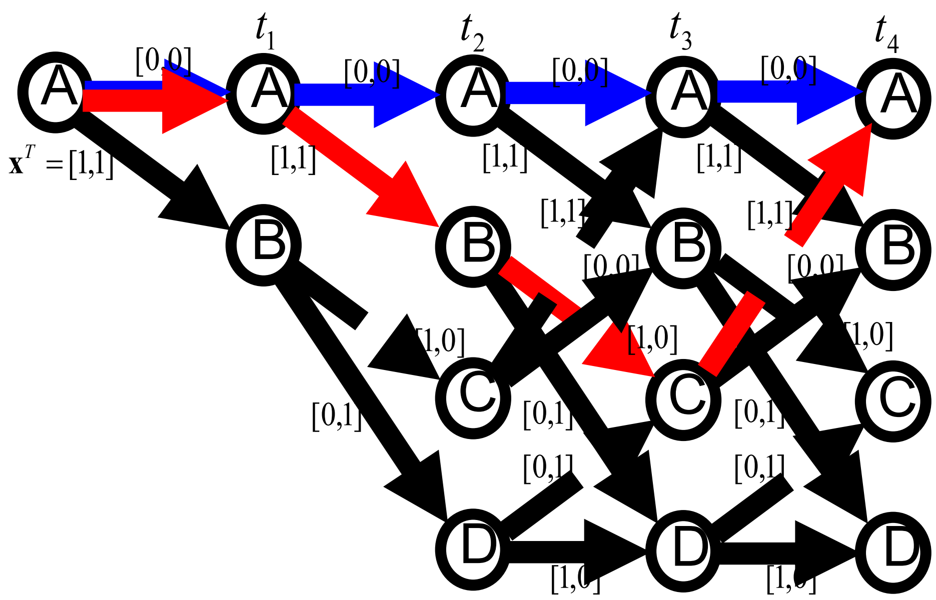
$$\mathbf{x}_2 = [00 \quad 11 \quad 10 \quad 11]$$

Pair-wise error probability

$$p_2(d) = \sum_{k=d_c+1}^{d} \binom{d}{k} p_e^k (1-p_e)^{d-k}$$

$$p_e = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\gamma}\right) \quad \text{if BPSK}$$

$$d_c = \left\lfloor \frac{d-1}{2} \right\rfloor$$



Code error probability

$$P_e < \sum_{d=d_{\min}}^{\infty} 2^{d-d_{\min}} P_2(d)$$

Puncture

Message

$m(1), m(2)$

Codes

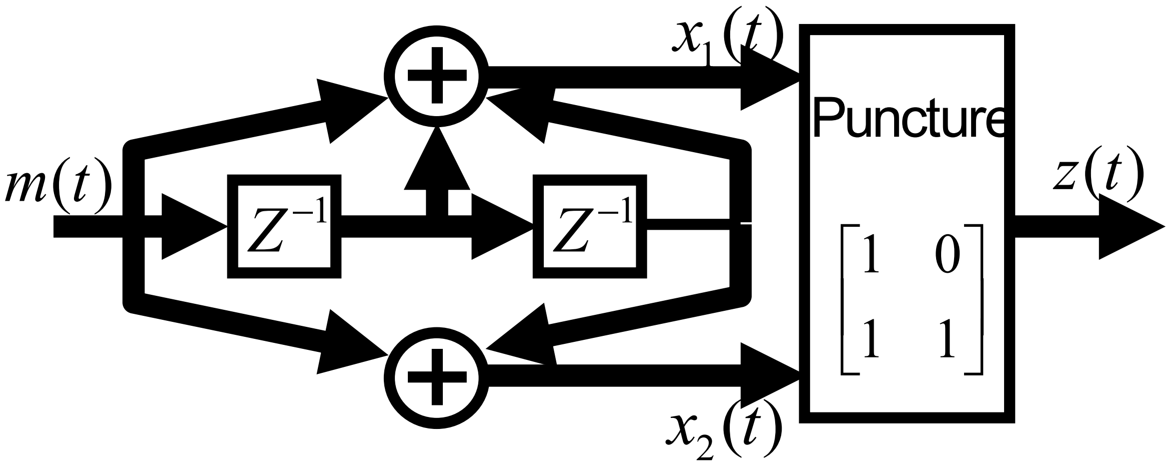
$x_1(1), x_1(2)$

$x_2(1), x_2(2)$

$r = 2/3$

Puncture

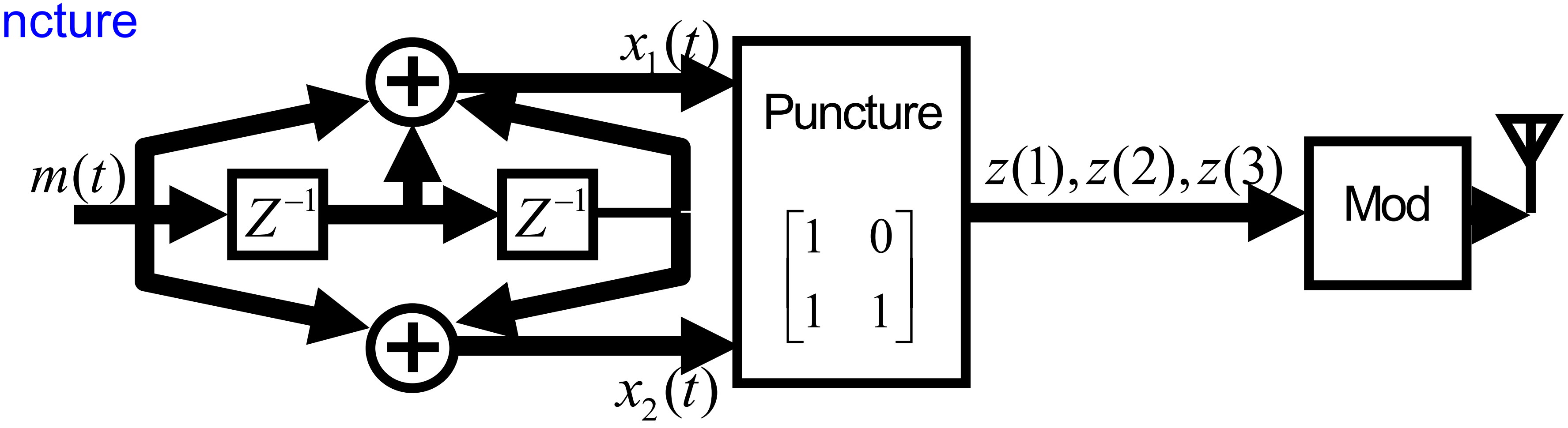
$z(1), z(2), z(3) = x_1(1), x_2(1), x_2(2)$



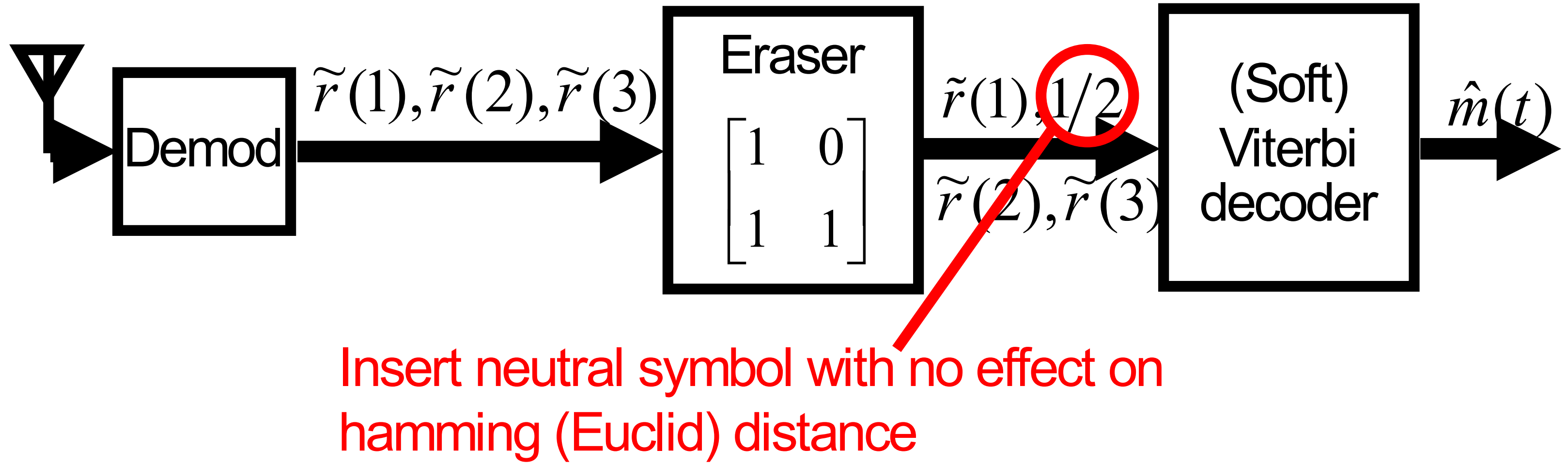
N	$r = 1/2$		$r = 2/3$		$r = 3/4$		$r = 4/5$	
	P	d_{min}	P	d_{min}	P	d_{min}	P	d_{min}
3	1 1	5	1 0 1 1	3	1 0 1 1 1 0	3	1 0 1 1 1 1 0 0	2

Eraser

Puncture

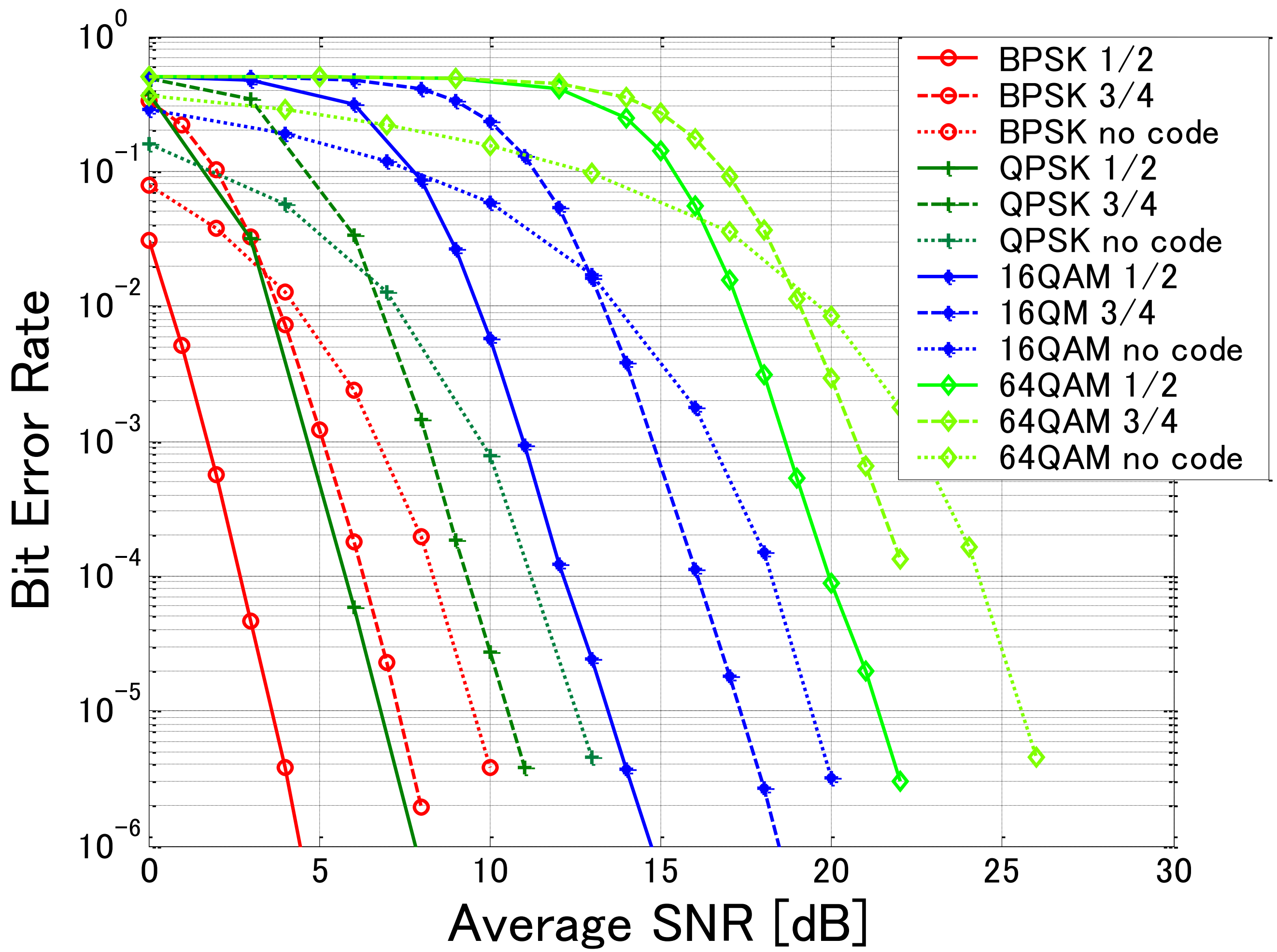


Eraser

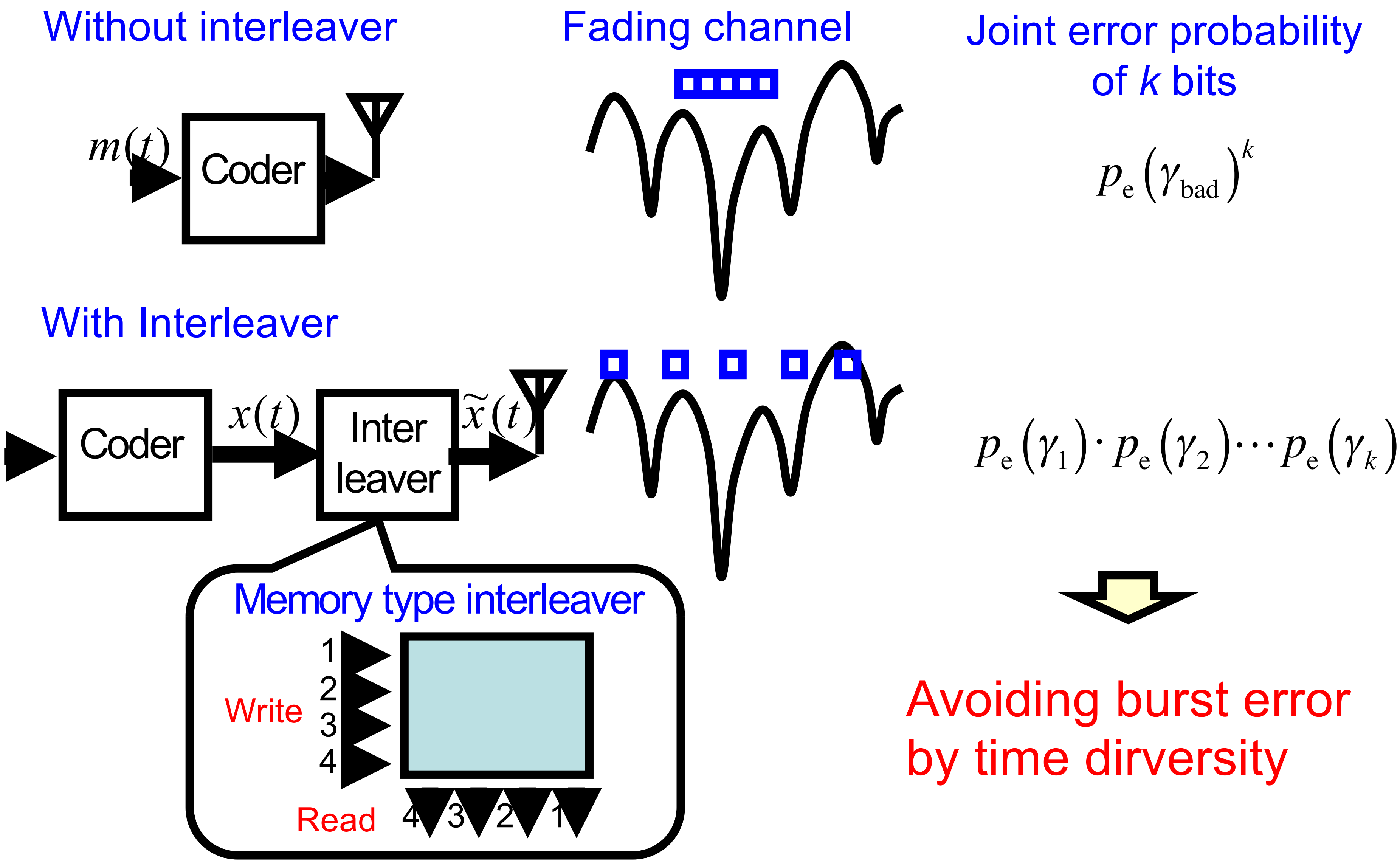


BER Performance

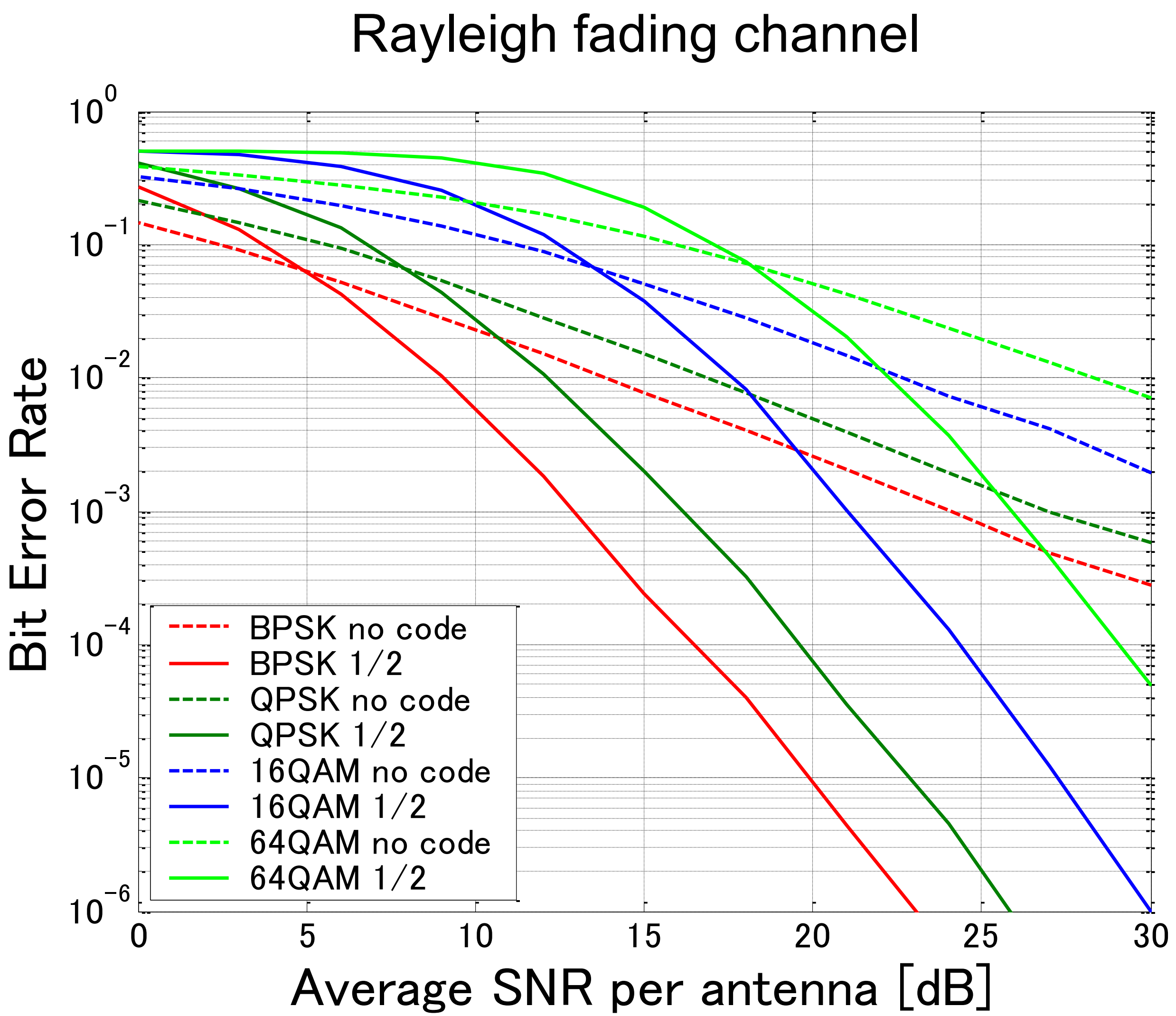
AWGN (Additive White Gaussian Noise) channel



Interleaver

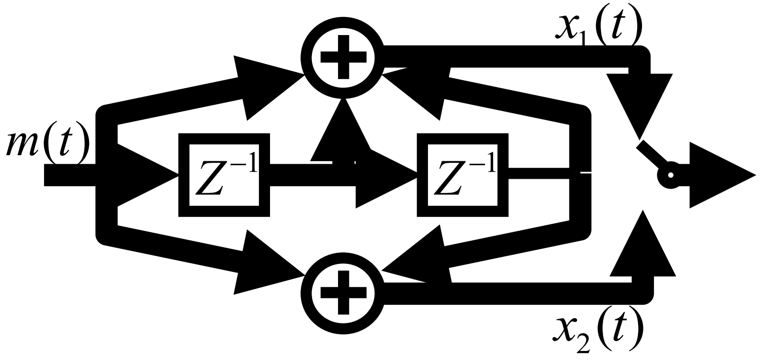


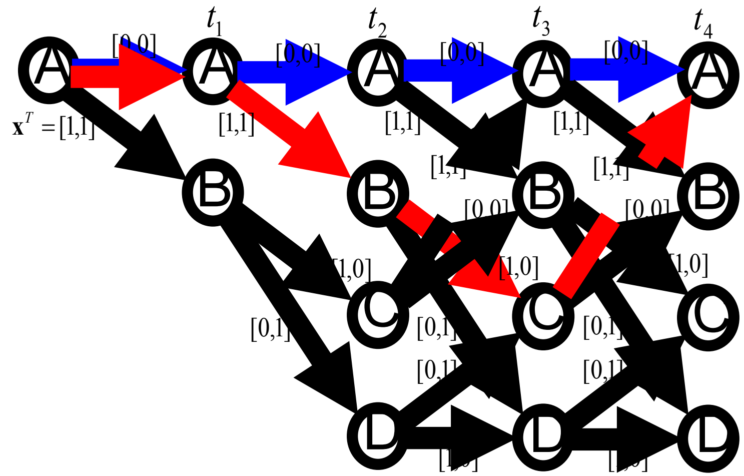
BER Performance in Fading Channel



Summary

- Convolutional coding & Viterbi decoding





- Error rate of Viterbi decoding

$$p_e < \sum_{d=d_{min}}^{\infty} 2^{d-d_{min}} p_2(d) \qquad p_2(d) = \sum_{k=d_c+1}^{d} \binom{d}{k} p_e^k (1-p_e)^{d-k}$$

- Interleaver & time diversity

Avoiding burst error $\quad p_e(\gamma_{bad})^k \Rightarrow p_e(\gamma_1) \cdot p_e(\gamma_2) \cdots p_e(\gamma_k)$

Viterbi Decoder (Soft Decision)

Receive signal

$$\tilde{\mathbf{r}}(t) = [\tilde{r}_1(t), \tilde{r}_2(t)]^T$$

Branch metric

$$b_{AA}(t) = \mathrm{euc}\left(\tilde{\mathbf{r}}(t), \tilde{\mathbf{x}}_{AA}\right)$$
$$b_{AC}(t) = \mathrm{euc}\left(\tilde{\mathbf{r}}(t), \tilde{\mathbf{x}}_{AC}\right)$$
$$\tilde{b}_{AA}(t) = \tilde{\mathbf{r}}(t) \cdot \tilde{\mathbf{x}}_{AA}$$
$$\tilde{b}_{AC}(t) = \tilde{\mathbf{r}}(t) \cdot \tilde{\mathbf{x}}_{AC}$$

Path metric

$$p_{AA}(t) = p_A(t-1) + \tilde{b}_{AA}(t)$$
$$p_{AC}(t) = p_C(t-1) + \tilde{b}_{AC}(t)$$

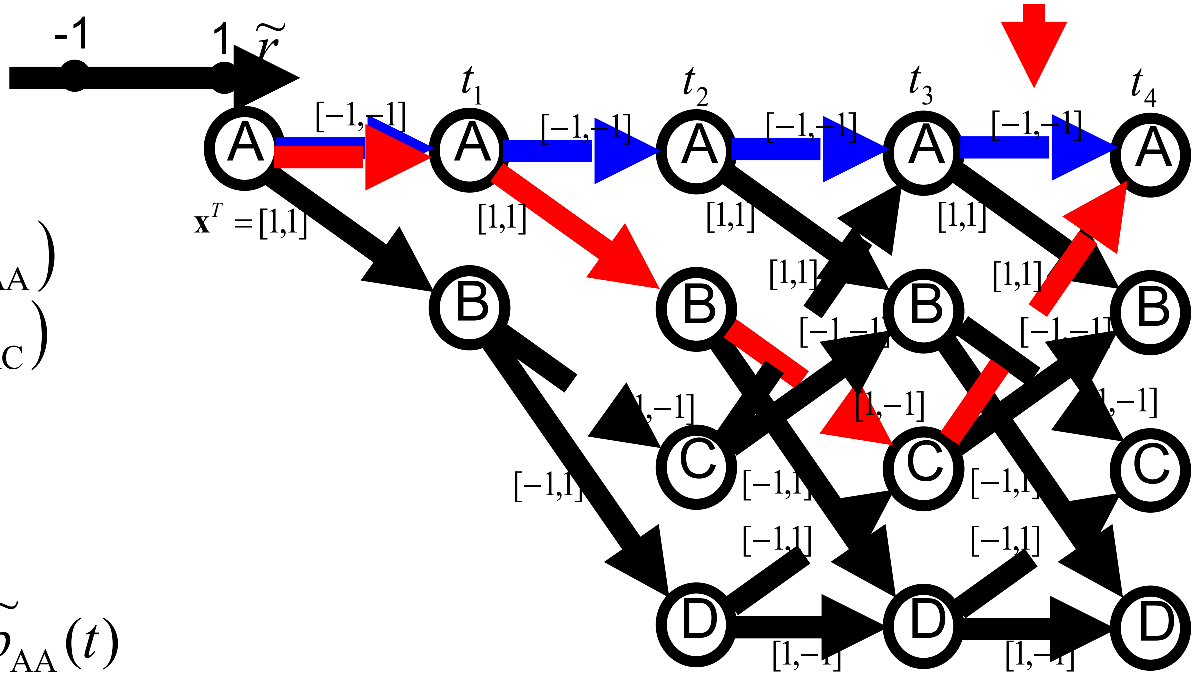
Survived path

$$p_A(t) = \max[p_{AA}(t), p_{AC}(t)]$$

Path decision

$$\hat{m}(t) = \arg\max[p_A(t), p_B(t), p_C(t), p_D(t)]$$

Decoded sequence



Metric of soft decision

$$\mathrm{euc}(\mathbf{a}, \mathbf{b}) = \sum_i \sqrt{|a_i - b_i|^2}$$
$$= \sum_i \sqrt{a_i^2 - 2a_i b_i + b_i^2}$$
$$\min[\mathrm{euc}(\mathbf{a}, \mathbf{b})] \equiv \max[\mathbf{a} \cdot \mathbf{b}]$$

Example of (Soft) Viterbi Decoding

Message sequence

Tail bits

$$\mathbf{m} = [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]$$

Coded sequence

$$\mathbf{x} = [-1 -1 \quad 1 1 \quad 1 -1 \quad 1 1 \quad -1 -1 \quad -1 -1]$$

Receive signal

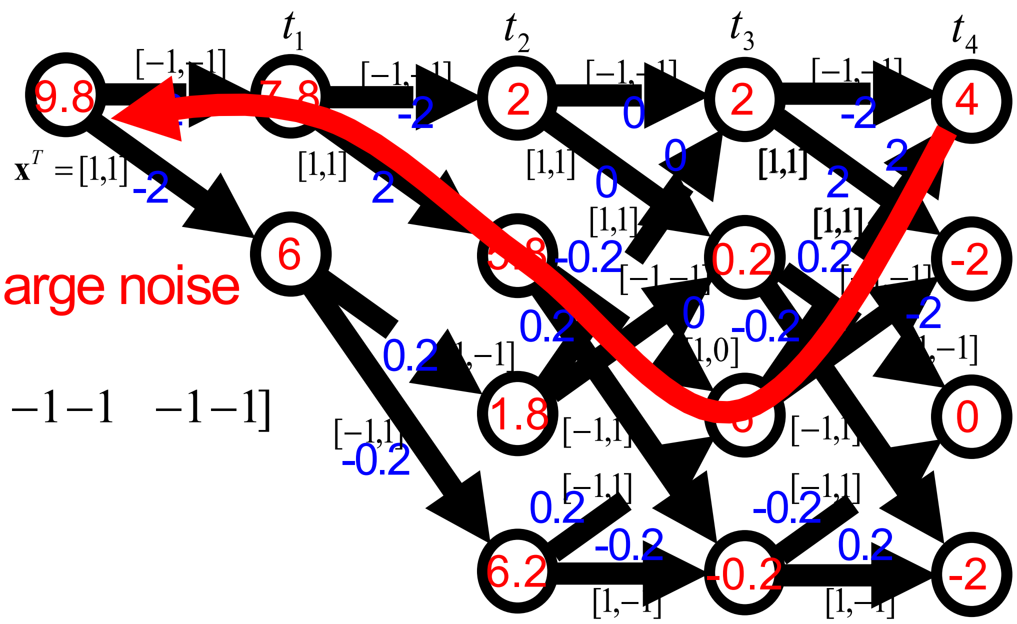
Symbol with large noise

$$\mathbf{r} = [-1.1 -0.9 \quad 1.1 0.9 \quad -0.1 0.1 \quad 1.1 0.9 \quad -1 -1 \quad -1 -1]$$

Decoded message

$$\hat{\mathbf{m}} = [0 \quad 1 \quad 0 \quad 0]$$

Soft decision Viterbi decoding



Error Rate of Soft Viterbi Decoding

Hamming distance

$$d = \mathrm{ham}(\mathbf{x}_1, \mathbf{x}_2)$$

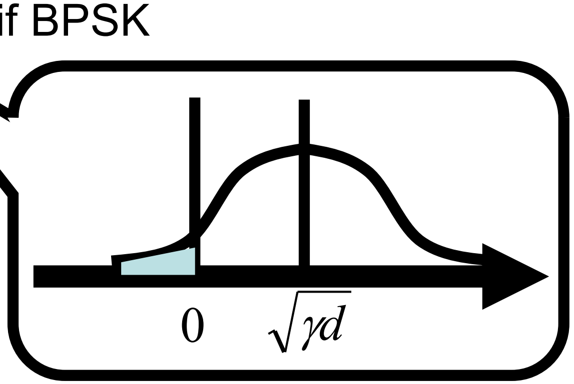
Pair-wise error probability

$$P_2(d) = P(p_2 > p_1)$$

$$= P\left(\sum_i \tilde{\mathbf{r}}(t_i)\cdot(\tilde{\mathbf{x}}_2(t_i) - \tilde{\mathbf{x}}_1(t_i)) > 0\right)$$

$$= P\left(\sum_{i=t_{\mathrm{diff}}} 2\tilde{\mathbf{r}}(t_i) > 0\right)$$

$$P_2(d) = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\gamma d}\right) \quad \text{if BPSK}$$



Code error rate

$$P_e < \sum_{d=d_{\min}}^{\infty} 2^{d-d_{\min}} P_2(d)$$

$$\mathbf{x}_1 = [00 \quad 00 \quad 00 \quad 00]$$

$$\tilde{\mathbf{x}}_1 = [-1 -1 \quad -1 -1 \quad -1 -1 \quad -1 -1]$$

$$\mathbf{x}_2 = [00 \quad 11 \quad 10 \quad 11]$$

$$\tilde{\mathbf{x}}_2 = [-1 -1 \quad 11 \quad 1 -1 \quad 11]$$