Neutron Transport Theory Lecture Note (6) - One-speed diffusion theory of a nuclear reactor (2) -

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5.2 The critical condition for general bare geometries

Considering a bare reactor of uniform composition surrounded by a free surface characterized by vacuum boundary conditions.

If the reactor is critical then the neutron flux must satisfy the steady-state diffusion equation.

$$-D\nabla^{2}\phi + \Sigma_{a}\phi(\mathbf{r}) = \nu\Sigma_{f}\phi(\mathbf{r}) \qquad \cdots (1)$$

boundary condition : $\phi(\tilde{\mathbf{r}}_s) = 0$ $\tilde{\mathbf{r}}_s$: extrapolated boundary

Dividing Eq.(1) by -D,

$$\nabla^2 \phi + \left(\frac{\nu \Sigma_{\rm f} - \Sigma_{\rm a}}{D}\right) \phi(\mathbf{r}) = 0 \qquad \cdots (2)$$

boundary condition : $\phi(\tilde{\mathbf{r}}_s) = 0$

i.e.
$$\nabla^2 \phi + \left(\frac{\mathbf{k}_{\infty} - 1}{\mathbf{L}^2}\right) \phi(\mathbf{r}) = 0$$
 ... (3)

boundary condition : $\phi(\tilde{\mathbf{r}}_s) = 0$

This equation is identical to that which generates the special eigenfunctions for this geometry.

$$\nabla^2 \psi_n + B_n^2 \psi_n(\mathbf{r}) = 0 \qquad \cdots (4)$$

boundary condition : $\psi(\mathbf{\tilde{r}}_{S}) = 0$

The requirement that the reactor is critical is the same as that for slab reactor,

$$B_{m}^{2} \equiv \left(\frac{\nu \Sigma_{f} - \Sigma_{a}}{D}\right) = B_{1}^{2} \equiv B_{g}^{2} \qquad \cdots (5)$$

The critical neutron flux distribution $\phi(\mathbf{r})$ is given by the fundamental eigenfunction $\psi_1(\mathbf{r})$.

Geometric buckling and flux profile for various bare core		
Bare core geometry	Geometric buckling	Flux profile
Slab (thickness :	$(\pi)^2$	πχ
a)	$\left(\frac{1}{\tilde{a}}\right)$	cos— ã
Sphere (radius :	$(\pi)^2$	r^{-1} cin (πr)
R)	$(\overline{\widetilde{R}})$	$1 \sin(\overline{\widetilde{R}})$
Rectangular	$(\pi)^2 + (\pi)^2 + (\pi)^2$	$\cos\left(\frac{\pi x}{x}\right)\cos\left(\frac{\pi x}{x}\right)\cos\left(\frac{\pi x}{x}\right)$
parallelepiped	$\left(\frac{\overline{a}}{\overline{a}}\right) + \left(\frac{\overline{b}}{\overline{b}}\right) + \left(\frac{\overline{c}}{\overline{c}}\right)$	$\cos\left(\frac{1}{\tilde{a}}\right)\cos\left(\frac{1}{\tilde{b}}\right)\cos\left(\frac{1}{\tilde{c}}\right)$

Eq.(4) will provide us with flux shape only in critical reactor.

The magnitude of neutron flux shall be determined by the total power P generated by the core.

$$P = \int_{v} d^{3}r w_{f} \Sigma_{f} \phi(\mathbf{r}) \qquad \cdots (6)$$

w_f : energy produced per fission event

5.3 Reflected reactor geometries

We consider a slab reactor with reflectors of nonmultiplying material of thickness of to the both side of the core.

 $Time\-independent\ diffusion\ equation\ \ (x{\geq}0)$

Core :
$$-D^{C}\frac{d^{2}\varphi^{C}}{dx^{2}} + (\Sigma_{a}^{C} - \nu\Sigma_{f}^{C})\varphi^{C}(x) = 0, \qquad 0 \le x \le \frac{a}{2} \qquad \cdots (7)$$

Reflector :
$$-D^R \frac{d^2 \phi^R}{dx^2} + \Sigma^R_a \phi^R(x) = 0, \qquad \frac{a}{2} \le x \le \frac{a}{2} + \tilde{b} \qquad \cdots (8)$$

Boundary conditions

(a)
$$\phi^{C}\left(\frac{a}{2}\right) = \phi^{R}\left(\frac{a}{2}\right)$$

(b) $J^{C}\left(\frac{a}{2}\right) = J^{R}\left(\frac{a}{2}\right)$... (9)
(c) $\phi^{R}\left(\frac{a}{2} + \tilde{b}\right) = 0$

General solution in the core (symmetric solution)

$$\Phi^{C}(x) = A^{C} \cos B_{m}^{C} x$$
where, $B_{m}^{C}{}^{2} \equiv \frac{\nu \Sigma_{f}^{C} - \Sigma_{a}^{C}}{D^{C}}$
...(10)

Solution in reflector which satisfies boundary condition (c)

$$\phi^{R}(x) = A^{R} \sinh\left[\frac{\frac{a}{2} + \tilde{b} - x}{L^{R}}\right] \qquad \cdots (11)$$
where, $L^{R} = \sqrt{\frac{D^{R}}{\Sigma^{R}}}$

where,
$$L^{R} = \sqrt{\frac{D^{R}}{\Sigma_{a}^{R}}}$$

By applying interface boundary conditions (a) and (b),

$$A^{C}\cos\left(\frac{B_{m}^{C}a}{2}\right) = A^{R}\sinh\left(\frac{\tilde{b}}{L^{R}}\right) \qquad \cdots (12)$$

$$D^{C}B_{m}^{C}A_{c}\sin\left(\frac{B_{m}^{C}a}{2}\right) = \frac{D^{R}}{L^{R}}A^{R}\cosh\left(\frac{\tilde{b}}{L^{R}}\right) \qquad \cdots (13)$$

Dividing Eq.(13) by Eq.(12),

$$D^{C}B_{m}^{C}\tan\left(\frac{B_{m}^{C}a}{2}\right) = \frac{D^{R}}{L^{R}}\coth\left(\frac{\tilde{b}}{L^{R}}\right) \qquad \cdots (14)$$

This equation is the reactor critical condition.

(cf.
$$B_m^2 = B_g^2$$
 in bare core)

Rewrite Eq.(14) as

$$\begin{pmatrix} \frac{B_{m}^{C}a}{2} \end{pmatrix} \tan\left(\frac{B_{m}^{c}a}{2}\right) = \frac{D^{R}a}{2D^{C}L^{R}} \coth\left(\frac{\tilde{b}}{L^{R}}\right) \qquad \cdots (15)$$

$$\frac{B_{m}^{C}a}{2} < \frac{\pi}{2} \quad \text{or} \quad B_{m}^{C}{}^{2} < \left(\frac{\pi}{a}\right)^{2}$$

$$\begin{bmatrix} \text{In bare (unreflected) core} \\ B_{m}{}^{2} = \left(\frac{\pi}{\tilde{a}}\right)^{2} \end{bmatrix}$$

It is conventional to define the difference between bare and reflected core dimensions as the reflector savings $\,\delta$:

$$\delta = [a(bare) - a(reflected)]/2 \qquad \cdots (16)$$

Ex. The reflector savings for the slab core

$$\delta = \frac{1}{B_{m}^{C}} \tan^{-1} \left[\frac{D^{C} B_{m}^{C} L^{R}}{D^{R}} \tanh\left(\frac{\tilde{b}}{L^{R}}\right) \right] \qquad \cdots (17)$$

For the thick reflector $\,b\gg L^R$

$$\delta \cong \frac{D^{C}}{D^{R}} L^{R} \qquad \cdots (18)$$

5.4 Reactor criticality calculations

(1) General procedure to determine geometries and material composition of critical reactors

Diffusion equation

$$\begin{split} &-\nabla D \nabla \varphi + \Sigma_a \varphi(\mathbf{r}) = \nu \Sigma_f \varphi(\mathbf{r}) & \cdots (19) \\ & \text{(no solutions in general unless the reactor is critical)} \\ & \text{boundary condition} \\ & \varphi(\tilde{\mathbf{r}}_s) = 0 \end{split}$$

We introduce on arbitrary parameter "k" into the equation.

$$-\nabla D\nabla \phi + \Sigma_{a} \phi(\mathbf{r}) = \frac{1}{k} \nu \Sigma_{f} \phi(\mathbf{r}) \qquad \cdots (20)$$

Picking up a core size and composition and solve the equation while determining k.

(eigenvalue problem)

k : multiplication eigenvalue

(2) Solution of eigenvalue problem by power method

Rewriting Eq.(20) in operator notation

$$M\phi = \frac{1}{k}F\phi \qquad \cdots (21)$$

where, $M \equiv -\nabla D(\mathbf{r})\nabla + \Sigma_a(\mathbf{r}) \equiv \text{Destruction operator}$ (leakage plus absorption) $F \equiv \nu \Sigma_f(\mathbf{r}) \equiv \text{Production operator}$ (fission)

Assuming the estimate $\phi^{(n)}$ and $k^{(n)}$ are given.

Estimate of fission source

$$S^{(n)} = F \phi^{(n)} \qquad \cdots (22)$$

We can iteratively solve for an improved source estimates $S^{(n+1)}$ from an earlier estimate $S^{(n)}$ by solving

$$M\phi^{(n+1)} = \frac{1}{k^{(n)}} S^{(n)}$$
 ... (23)

for $\phi^{(n+1)}$ and then computing

$$S^{(n+1)} = F\varphi^{(n+1)} \qquad \cdots (24)$$

as n becomes large, $\phi^{(n+1)}$ will converge to the true eigenfunction $\phi(\mathbf{r})$ that satisfies Eq.(21) with the eigenvalue

$$\mathsf{M}\phi^{(n+1)} \cong \frac{1}{k^{(n+1)}} \mathsf{F}\phi^{(n+1)} \tag{25}$$

If we integrate Eq.(25) overall space, we should be able to obtain a resonance estimate for $k^{(n+1)}$ as

$$\mathbf{k}^{(n+1)} \cong \frac{\int d^3 \mathbf{r} \mathbf{F} \boldsymbol{\Phi}^{(n+1)}}{\int d^3 \mathbf{r} \mathbf{M} \boldsymbol{\Phi}^{(n+1)}} \qquad \cdots (26)$$

From Eq.(23), Eq.(24)

This shows the eigenvalue, k in Eq.(21) is the same as the effective multiplication factor, that is the ratio of the number of neutrons in two consecutive fission generations in the reactor.

In $k\neq 1$, to make the reactor critical, we can change the reactor size and composition and repeat the calculation.