

Neutron Transport Theory Lecture Note (4)
-One-Speed diffusion Theory-

Toru Obara
Tokyo Institute of Technology

4. The one-speed diffusion equation

4.1 Derivation of the diffusion equation

We will characterize the neutron distribution in the reactor by the neutron density $N(\mathbf{r}, t)$ (or by the neutron flux $\phi(\mathbf{r}, t)$)

We consider on arbitrary volume V of surface area S located anywhere within the reactor.

The total number of neutron in V at time t

$$\int_V d^3r N(\mathbf{r}, t) = \int_V d^3r \frac{1}{v} \phi(\mathbf{r}, t) \quad \dots (1)$$

The time rate of change of the number of neutron in V

$$\begin{aligned} \frac{d}{dt} \left[\int_V d^3r \frac{1}{v} \phi(\mathbf{r}, t) \right] &= \int_V d^3r \frac{1}{v} \frac{\partial \phi}{\partial t} \\ &= \text{Production in } V - \text{Absorption in } V - \text{Net leakage from } V \end{aligned} \quad \dots (2)$$

$$\text{Production in } V = \int_V d^3r S(\mathbf{r}, t) \quad \dots (3)$$

$S(\mathbf{r}, t)$: neutron source density

$$\text{Asorption in } V = \int_V d^3r \Sigma_a(\mathbf{r}) \phi(\mathbf{r}, t) \quad \dots (4)$$

$$\text{Net leakage from } V = \int_S d\mathbf{S} \cdot \mathbf{J}(\mathbf{r}, t) \quad \dots (5)$$

Convert the surface integral into volume integral by using Gauss's theory

$$\int_S d\mathbf{S} \cdot \mathbf{J}(\mathbf{r}, t) = \int_V d^3r \nabla \cdot \mathbf{J}(\mathbf{r}, t) \quad \dots (6)$$

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) = \text{div} \mathbf{J}(\mathbf{r}, t)$$

Substituting each of these mathematical expressions into Eq.(2),

$$\int_V d^3r \left[\frac{1}{v} \frac{\partial \phi}{\partial t} - S + \Sigma_a \phi + \nabla \cdot \mathbf{J} \right] = 0 \quad \dots (7)$$

Eq.(7) must hold for any volume V

$$\frac{1}{v} \frac{\partial \phi}{\partial t} - S + \Sigma_a \phi + \nabla \cdot \mathbf{J} = 0 \quad \dots (8)$$

The equation contains two unknowns $\phi(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$. There is no exact relationship between $\phi(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$.

Diffusion approximation

$$\mathbf{J}(\mathbf{r}, t) \cong -D(\mathbf{r}) \nabla \phi(\mathbf{r}, t) \quad \dots (9)$$

where

$$D = \frac{1}{3\Sigma_{tr}} = \frac{1}{3(\Sigma_t - \bar{\mu}_0 \Sigma_s)} \quad \dots (10)$$

(Diffusion coefficient)

$\bar{\mu}_0$: The average cosine of the scattering angle in a neutron scattering collision

Substituting Eq.(9) into Eq.(8),

$$\frac{1}{v} \frac{\partial \phi}{\partial t} = \nabla \cdot D(\mathbf{r}) \nabla \phi(\mathbf{r}) - \Sigma_a(\mathbf{r}) \phi(\mathbf{r}, t) + S(\mathbf{r}, t) \quad \dots (11)$$

(One-speed neutron diffusion equation)

If the D and Σ_a do not depend on position (homogeneous)

$$\frac{1}{v} \frac{\partial \phi}{\partial t} - D \nabla^2 \phi + \Sigma_a \phi(\mathbf{r}, t) = S(\mathbf{r}, t) \quad \dots (12)$$

If the flux is not function of time (steady state)

$$-D \nabla^2 \phi + \Sigma_a \phi(\mathbf{r}) = S(\mathbf{r}) \quad \dots (13)$$

Dividing by $-D$

$$\nabla^2 \phi(\mathbf{r}) - \frac{1}{L^2} \phi(\mathbf{r}) = -\frac{S(\mathbf{r})}{D} \quad \dots (14)$$

where

$$L \equiv \sqrt{\frac{D}{\Sigma_a}} \quad (\text{Diffusion length})$$

L : A measure of how far the neutrons will diffuse from a source before they are absorbed.

[Important]

- The conditions that the diffusion approximation can be valid
 - ① It is used to describe the neutron flux several mean free path away from the boundaries or isolated source
 - ② The medium is only weakly absorbing
 - ③ The neutron current is changing slowly on a time scale comparable to the mean time between neutron-nuclei collision

4.2 Initial and Boundary Conditions

(1) Initial condition

Specifying the neutron flux $\phi(\mathbf{r}, 0)$ for all positions \mathbf{r} at the initial time $t=0$

Initial condition : $\phi(\mathbf{r}, 0) = \phi_0(\mathbf{r})$

(2) Boundary conditions

(a) Vacuum boundary

Ex. Outside boundary of a reactor

No neutrons can enter the reactor through this surface from outside.

A boundary condition that give correct neutron flux deep within the reactor where diffusion theory is valid is :

$$\phi(\tilde{x}_s) = 0$$

$$\tilde{x}_s = x_s + z_0 \quad (\text{extrapolated boundary})$$

where

$$z_0 = 0.7104 \lambda_{tr} \quad (\text{extrapolation length})$$

$$\lambda_{tr} = \frac{1}{\Sigma_{tr}}$$

(b) Boundary condition for interfaces (material discontinuity)

- ① Continuity of the neutron flux $\phi(\mathbf{r}, t)$
- ② Continuity of the normal component of the neutron current density $\mathbf{J}(\mathbf{r}, t)$

(c) The condition which should be satisfied always

$$0 \leq \phi(\mathbf{r}, t) < \infty \quad (\text{except in the neighborhood of localized sources})$$

4.3 Neutron diffusion in nonmultiplying media

Example: Neutron flux $\phi(x)$ in an infinite homogeneous medium with plane source at the origin

Diffusion equation

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0, \quad x > 0$$

Boundary conditions

$$(a) \lim_{x \rightarrow 0^+} -D \frac{d\phi}{dx} = \frac{S_0}{2} \quad S_0 : \text{neutron source} [s^{-1} \cdot \text{cm}^{-2}]$$

$$(b) \lim_{x \rightarrow \infty} \phi(x) < \infty$$

The general solution

$$\phi(x) = A \exp\left(-\frac{x}{L}\right) + B \exp\left(\frac{x}{L}\right)$$

From boundary condition (b),

$$B = 0$$

From boundary condition (a),

$$\lim_{x \rightarrow 0^+} -D \left(-\frac{A}{L} \exp\left(-\frac{x}{L}\right) \right) = \frac{AD}{L} = \frac{S_0}{2}$$

$$\therefore A = \frac{S_0 L}{2D}$$

$$\text{Hence } \phi(x) = \frac{S_0 L}{2D} \exp\left(-\frac{x}{L}\right), \quad x > 0$$

By symmetry

$$\phi(x) = \frac{S_0 L}{2D} \exp\left(\frac{x}{L}\right), \quad x < 0$$