The First Example of Endogenous Growth: The Role of Learning-by-Doing Externalities

IEE.B402. Advanced Macroeconomics

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Plan

- Model setup
- Households' and firms' behavior
- Specification of technology level
  $\rightarrow$ Property of aggregate production function
- Characterization of the competitive equilibrium path
- Intuitions behind the obtained results

Setup

- Time is continuous, denoted by $t \in [0, \infty)$
- Two types of economic agents: households and firms
- Perfect competition
- Population is fixed at $\bar{L} > 0$ over time, that is, there is no population growth (i.e., $n = 0$)

$(*)$ The notations are basically same as the Ramsey–Cass–Koopmans model.

Households' Behavior: Utility Maximization Problem

- Households are homogeneous.
- A representative household's dynamic utility maximization problem:

$$\max_{\{c(t),a(t)\}} \quad U = \int_0^\infty e^{-\rho t} u(c(t))dt$$

$$\text{s.t.} \quad \dot{a}(t) = r(t)a(t) + w(t) - c(t) \qquad \forall t \in [0,\infty) \tag{1}$$

$$\lim_{t\to\infty} a(t)\exp\left(-\int_0^t r(s)ds\right) \geq 0 \tag{2}$$

$$a(0) \text{ given}$$

- Specification of $u(c)$ as the CRRA form:

$$u(c) = \begin{cases} \dfrac{c^{1-\theta}-1}{1-\theta} & \text{if } \theta > 0, \theta \neq 1, \\ \ln c & \text{if } \theta = 1. \end{cases}$$

Households' Behavior: Euler equation and TVC

- Current-value Hamiltonian: $H(a, c, \lambda) = u(c) + \lambda[ra + w - c]$
- First-order-conditions (F.O.Cs):

$$\frac{\partial H}{\partial c(t)} = 0 : \quad c(t)^{-\theta} = \lambda(t) \tag{3}$$

$$\dot{\lambda}(t) = \rho\lambda(t) - \frac{\partial H}{\partial a(t)} : \quad \dot{\lambda}(t) = (\rho - r(t))\lambda(t) \tag{4}$$

Eqs. (3) and (4) lead the Euler equation:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho) \tag{5}$$

- Transversality condition (TVC) $\rightarrow$ the binding condition of (2):

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t r(s)ds\right) = 0 \tag{6}$$

Firms' Behavior: Production Technology

- There are M firms, indexed by i.
  - $i \in \{1, 2, \dots, M\}$ if we assume firms are discrete; or
  - $i \in [0, M]$ if we assume firms are continuous
- Production function of firm i:

$$Y_i(t) = K_i(t)^\alpha (A(t)L_i(t))^{1-\alpha}, \quad \alpha \in (0,1) \tag{7}$$

  - $Y_i(t)$, $K_i(t)$, $L_i(t)$: output, capital input, and labor input by firm i
  - $A(t)$: technology level, common across firms.

($*$) The Cobb–Douglas specification is just for simplicity.
We can obtain the same results obtained below even under more general form of production function.

Firms' Behavior: Profit Maximization

- Let $k_i(t) \equiv K_i(t)/L_i(t)$ denote the "capital-labor ratio" of firm i.
- Profit maximization problem of firm i:

$$\max_{k_i(t), L_i(t)} \left[k_i(t)^\alpha A(t)^{1-\alpha} - R(t)k_i(t) - w(t) \right] L_i(t)$$

  with taking $A(t)$ as given.

  - $R(t)$: the rental price of capital, satisfying $R(t) - \delta = r(t)$

- F.O.Cs of profit maximization:

$$R(t) = \alpha \left(\frac{k_i(t)}{A(t)} \right)^{-(1-\alpha)} \tag{8}$$

$$w(t) = (1-\alpha) k_i(t)^\alpha A(t)^{1-\alpha} \tag{9}$$

  Thus, all firms determine the same capital-labor ratio.

Specification of $A(t)$

- Let $K(t)$ denote the aggregate stock of capital at time t:

$$K(t) = \int_0^M K_i(t)di$$

- Specification of technology level $A(t)$ in Romer (1986):

$$A(t) = zK(t), \quad z > 0 \tag{10}$$

  $\leftrightarrow$ Although each firm takes $A(t)$ as given, this technology advances endogenously for the economy as a whole.

- Such a specification is motivated by "learning-by-doing (LBD)" externalities.

Property of Aggregate Increasing Returns to Scale

- Let $L(t) = \int_0^M L_i(t)di$ denote the aggregate labor demand.
- Since $k_i(t)$ are same across firms, we easily have

$$K(t)/L(t) = k_i(t) \forall i \tag{11}$$

- Then, the aggregate output (denoted by $Y(t)$) is given by

$$\begin{aligned} Y(t) &= \int_0^M Y_i(t)di = \int_0^M k_i(t)^\alpha A(t)^{1-\alpha} L_i(t)di \\ &= \left(\frac{K(t)}{L(t)}\right)^\alpha A(t)^{1-\alpha} \int_0^M L_i(t)di \\ &= K(t)^\alpha A(t)^{1-\alpha} L(t)^{1-\alpha} \\ &= K(t)(zL(t))^{1-\alpha} \quad (\because \text{Eq.}(10)) \end{aligned}$$

If K and L are doubled, Y becomes more than doubled.

Characterization of the competitive equilibrium path

Market-Clearing Conditions

- Market-clearing conditions:

$$\text{Labor market:} \quad \bar{L} = L(t) \tag{12}$$

$$\text{Asset market:} \quad a(t)\bar{L} = K(t) \tag{13}$$

- Substituting (10)-(12) into (8) and (9), we obtain

$$R(t) = \alpha\bar{A} \tag{14}$$

$$w(t) = (1-\alpha)\bar{A}K(t)/\bar{L} \tag{15}$$

where $\bar{A} \equiv (z\bar{L})^{1-\alpha}$.

- From (14) and (15), we can easily verify that

$$R(t)K(t) + w(t)\bar{L} = Y(t)(\equiv \bar{A}K(t)) \tag{16}$$

Dynamic System

- Using (1), (13), (16) and $r(t) = R(t) - \delta$, we obtain the dynamics of $K(t)$ as follows:

$$\dot{K}(t) = (\bar{A} - \delta)K(t) - C(t) \quad (\text{where } C(t) \equiv c(t)\bar{L}) \tag{17}$$

- Substituting (14) and $r(t) = R(t) - \delta$ into the Euler equation (5), we obtain the following dynamics of $C(t)$:

$$\frac{\dot{C}(t)}{C(t)} = \underbrace{\frac{1}{\theta}(\alpha\bar{A} - \delta - \rho)}_{\equiv\gamma} \tag{18}$$

- Since $r(t) = R(t) - \delta = \alpha\bar{A} - \delta$, TVC (6) is rewritten as

$$\lim_{t\to\infty} K(t) \exp\left[-\left(\alpha\bar{A} - \delta\right)t\right] = 0 \tag{19}$$

Reduced Dynamic System

- Eqs. (17)–(19) jointly constitute the dynamic system.
- If we define $x(t) \equiv C(t)/K(t)$, from (17) and (18) we obtain

$$\begin{aligned}\frac{\dot{x}(t)}{x(t)} &= \frac{\dot{C}(t)}{C(t)} - \frac{\dot{K}(t)}{K(t)} \\ &= \frac{1}{\theta}(\alpha\bar{A} - \delta - \rho) - \left(\bar{A} - \delta - x(t)\right) \\ &= x(t) - \underbrace{\frac{1}{\theta}\left[\rho + (\theta - \alpha)\bar{A} - (\theta - 1)\delta\right]}_{\equiv\varphi} \qquad (20)\end{aligned}$$

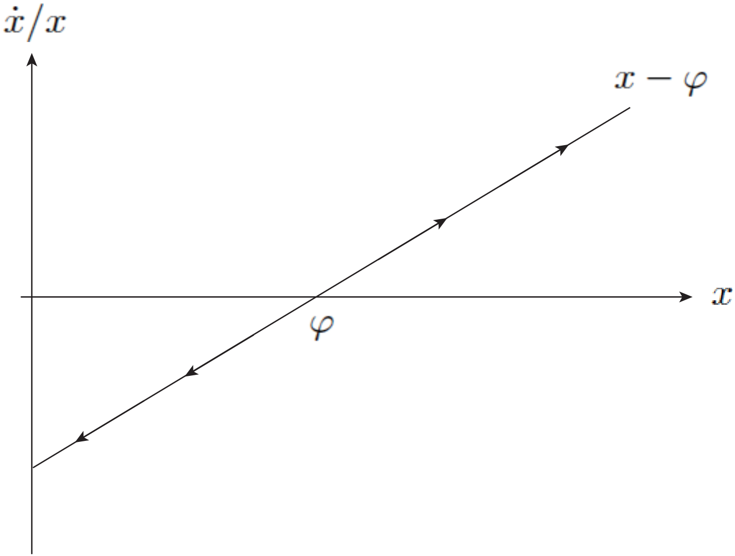
Assumption 1

$\varphi > 0$

Assumption 2

$\gamma > 0$, *that is,* $\alpha\bar{A} - \delta - \rho > 0$

Phase Diagram

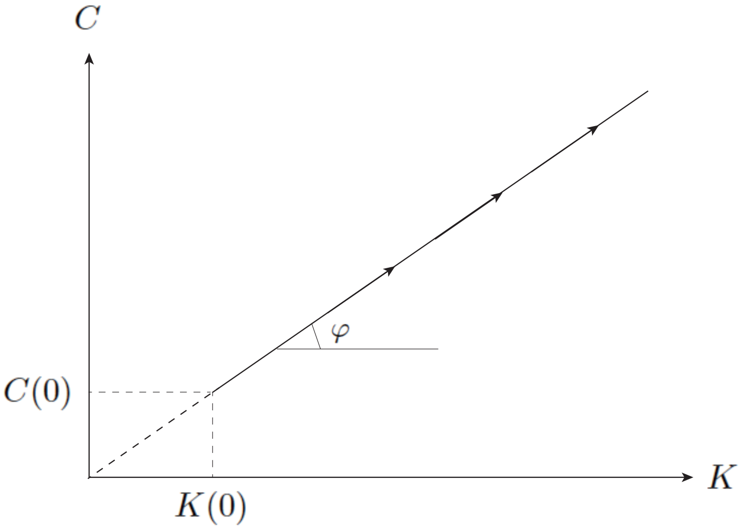


Uniqueness of Equilibrium Path

- Note that $C(0)$ is an endogenous variable, while $K(0)$ is not.
  $\rightarrow x(0)$ is an endogenous variable, determined in the model.

- Determination of $x(0)$:
  1. If $x(0) > \varphi$, then $\{x(t)\}$ diverges to infinity.
     $\rightarrow$ Such a path is not an equilibrium, because this is not feasible.
  2. If $x(0) < \varphi$, then $\{x(t)\}$ eventually becomes zero.
     $\rightarrow$ Such a path is not an equilibrium, because this violates the TVC.
  3. If $x(0) = \varphi$, then $x(t) = \varphi \ \ \forall t \in [0, \infty)$.
     $\therefore \dot{K}(t)/K(t) = \dot{C}(t)/C(t) = \gamma > 0 \ \forall t \in [0, \infty)$

Sustained and Balanced Growth



Check of TVC

- Finally, we have to check that this balanced growth path is consistent with TVC.
- Since $\dot{K}(t)/K(t) = \gamma > 0 \forall t$, $K(t) = K(0)\exp(\gamma t)$. Substituting this into the left-hand-side (LHS) of TVC:

$$\begin{aligned}\text{LHS} &= K(0) \lim_{t \to \infty} \exp\left[\left(\gamma - \alpha \bar{A} + \delta\right) t\right] \\ &= K(0) \lim_{t \to \infty} \exp(-\psi t) \quad (\because \text{Definitions of } \gamma \text{ and } \varphi)\end{aligned}$$

where

$$\psi = \frac{1}{\theta}\left(\rho + (\theta - 1)(\alpha \bar{A} - \delta)\right) < \varphi$$

For the TVC to be satisfied, $\psi > 0$ must be imposed.

Assumption 3

$\psi > 0$

Equilibirum

($*$) If we impose Assumption 3, then Assumption 1 is automatically satisfied.

$\downarrow$

Proposition 1

Suppose that Assumptions 2 and 3 are satisfied. Then

1. *There is the unique competitive equilibrium path.*
2. *There is no transitional dynamics.*
3. *GDP, consumption, capital, and wage income grow at the same positive rate of* $\gamma > 0$ *from the initial date.*

Implications

- Paul Romer's seminal paper (Romer, 1986)

  $\rightarrow$ LBD externalities induces a technological change.

  $(*)$ The work of Arrow (1962) formed the basis for this research.

  $\downarrow$
- The growth rate γ now depends on
  - Preference parameters (ρ, θ)
  - Technology parameters (z, α, δ)
  - Population size $(\bar{L})$
- Furthermore, if we introduce the taxation capital income, we can find that such a taxation harms economic growth.
- See Barro and Sala-i-Martin (2004, Ch.4) and Acemoglu (2009, Ch. 11) for further discussions on the LBD externalities.

References

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- Arrow, K. J. (1962) "The Economic Implications of Learning by Doing." *Review of Economic Studies* 29, pp. 155–173.
- Barro, R. J. and X. Sala-i-Martin (2004) *Economic Growth*, Second Edition, Cambridge, MIT Press.
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