Mathematical Preliminary (2) Infinite-Horizon Dynamic Optimization in Continuous Time

IEE.B402. Advanced Macroeconomics

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Plan

- Infinite-horizon utility maximization problem
  - Two key conditions:
    Euler equation and the transversality condition (TVC)
  - Sufficiency of Euler eq. and TVC for utility maximization
- How to derive the utility-maximizing conditions
  - Hamiltonian approach (based on Maximum Principle)
- Economic implications of dynamic utility maximization
  - Intertemporal budget constraint
  - Consumption function with utility maximization
  - Role of expectations about the "future" in the "current" decision making

($*$) The detail of derivations for key equations is given in the class.

Notation

Notation:

- Time is continuous, indexed by $t \in [0, \infty)$.
- As for the other variables, they have the same meanings as the slides on Jun. 19, but they are now continuous functions of time.
- Hereafter, let $\{x(t)\}$ denote the time path of a variable $x(t)$.

Budget constraint at time t:

$$\dot{a}(t) \equiv \frac{da(t)}{dt} = r(t)w(t) + w(t) - c(t). \tag{1}$$

A dot over a variable indicates its time derivative (e.g., $\dot{x}(t) = dx(t)/dt$).

Time Discounting in Continuous Time

Let $\mathcal{D}(t) \in (0, 1)$ denote the discount factor from time 0 to t, and assume that

$$\mathcal{D}(t) = \mathcal{D}(t - \Delta t) \times \frac{1}{1 + \rho(t)\Delta t},$$

- $\rho(t) > 0$: the discount rate from time $t - \Delta t$ to t.

Taking the limit of $\Delta t \to 0$ yields

$$\dot{\mathcal{D}}(t) = -\rho(t)\mathcal{D}(t) \;\Rightarrow\; \mathcal{D}(t) = \exp\left(-\int_0^t \rho(s)ds\right).$$

- If $\rho(t)$ is constant over time (i.e., $\rho(t) = \rho$), the discount factor is simply given by $\mathcal{D}(t) = e^{-\rho t}$.

Note

Hereafter we use "$\exp(-\rho t)$" and "$e^{-\rho t}$" interchangeably.

Objective Function

Suppose that a household has the constant discount rate $\rho > 0$.
Objective function:

$$U = \int_0^\infty e^{-\rho t} u(c(t)) dt,$$

where $u(\cdot)$ is called the instantaneous utility function.

Assumption 1

1. *u is differentiable (of necessary times);*
2. *u is strictly increasing: $u'(c) > 0$;*
3. *u is strictly concave: $u''(c) < 0$.*

Assumption 2

The function u satisfies the Inada condition: $\lim_{c \to 0^+} u'(c) = \infty$.

Continuous-Time Utility Maximization Problem

The canonical utility maximization problem in continuous time is formulated as

$$\max_{\{c(t),a(t)\}} \quad U = \int_0^\infty e^{-\rho t} u(c(t))dt$$

$$\text{s.t.} \quad \dot{a}(t) = r(t)w(t) + w(t) - c(t) \ \forall t \in [0, \infty), \tag{1}$$

$$\lim_{t\to\infty} a(t) \exp\left(- \int_0^t r(s)ds \right) \geq 0. \tag{NPG}$$

A pair of paths $\{c(t), a(t)\}$ is called feasible or admissible if it satisfies (1) and (NPG).

Sufficient Conditions of UMP

Theorem 1 (Sufficiency)

Suppose that $u(c)$ satisfies Assumption 1. Then, given $a(0)$ and $\{r(t), w(t)\}$, the pair of time paths $\{c(t), a(t)\}$ is the utility-maximizing plan if it satisfies (1) and

$$\sigma(c(t))\frac{\dot{c}(t)}{c(t)} = r(t) - \rho, \tag{2}$$

$$\lim_{t\to\infty} e^{-\rho t}u'(c(t))a(t) = 0, \tag{3}$$

where $\sigma(c) = -\frac{cu''(c)}{u'(c)} > 0$ is the degree of relative risk aversion.

Proof. Appendix.

Note:

(2) is the continuous-time counterpart of the Euler equation, and

(3) is that of the transversality condition (TVC).

How to derive the conditions

Hamiltonian Approach

1. Construct the Lagrangian:

$$L = \int_0^\infty e^{-\rho t} \left\{ u(c(t)) + \lambda(t) \left[r(t)a(t) + w(t) - c(t) - \dot{a}(t) \right] \right\} dt,$$

   where $\lambda(t)$ is the Lagrangian multiplier at time t.

2. Arrange:

$$\begin{aligned} L &= \int_0^\infty e^{-\rho t} \left\{ u(c) + \lambda \left[ra + w - c \right] \right\} dt - \int_0^\infty e^{-\rho t} \lambda(t)\dot{a}(t) dt \\ &= \int_0^\infty e^{-\rho t} H(a(t), c(t), \lambda(t), t) dt - \int_0^\infty e^{-\rho t} \lambda(t)\dot{a}(t) dt, \end{aligned}$$

   where

$$H(a(t), c(t), \lambda(t), t) = u(c(t)) + \lambda(t)[r(t)a(t) + w(t) - c(t)],$$

   is called the current-value Hamiltonian.

Hamiltonian Approach

3. Using the method of integration by parts, derive

$$\int_0^\infty e^{-\rho t}\lambda(t)\dot{a}(t)dt = -\lambda(0)a(0) + \lim_{t\to\infty} e^{-\rho t}\lambda(t)a(t) - \int_0^\infty e^{-\rho t}a(t)\left[\dot{\lambda}(t) - \rho\lambda(t)\right]dt,$$

which results in

$$L = \int_0^\infty e^{-\rho t}\left\{H(a(t), c(t), \lambda(t), t) + a(t)\left[\dot{\lambda}(t) - \rho\lambda(t)\right]\right\}dt + \lambda(0)a(0) - \lim_{t\to\infty} e^{-\rho t}\lambda(t)a(t).$$

Hamiltonian Approach

4. Then, derive the first-order (necessary) conditions wrt $c(t)$ and $a(t)$:

$$c(t): \quad \frac{\partial H(\cdot)}{\partial c(t)} = 0 \Leftrightarrow u'(c(t)) - \lambda(t) = 0, \tag{4}$$

$$a(t): \quad \frac{\partial H(\cdot)}{\partial a(t)} + \dot{\lambda}(t) - \rho\lambda(t) = 0 \Leftrightarrow \dot{\lambda}(t) = (\rho - r(t))\lambda(t). \tag{5}$$

5. Take the log of both sides in (4) and differentiate the results wrt time:

$$\frac{cu''(c)}{u'(c)}\frac{\dot{c}}{c} = \frac{\dot{\lambda}}{\lambda}.$$

6. Finally, by substituting this equation into (5), we obtain the Euler equation:

$$\sigma(c(t))\frac{\dot{c}(t)}{c(t)} = r(t) - \rho.$$

Relationship btw NPG and TVC

Using (5), we have

$$\lambda(t) = \lambda(0) \exp\left(-\int_0^t (r(s) - \rho) ds \right). \tag{6}$$

Since $\lambda(t) = u'(c(t))$ from (4), the TVC (3) is rewritten as

$$\lim_{t \to \infty} a(t) \exp\left(-\int_0^t r(s) ds \right) = 0.$$

$\Downarrow$

- NPG is the constraint, which prohibits the household to leave a debt in a present value sense.
- TVC is the condition for maximization, which ensures the binding of the NPG.

Economic implications of dynamic utility maximization

Implications

In a two-period utility maximization problem, $\{c_0, c_1\}$ is obtained from

- Intertemporal budget constraint:

$$(1 + r_0)a_0 + w_0 + \frac{w_1}{1 + r_1} = c_0 + \frac{c_1}{1 + r_1}, \tag{7}$$

  and

- Euler equation (in discrete time):

$$u'(c_0) = \beta(1 + r_1)u'(c_1), \tag{8}$$

  where $r(t)$ and $w(t)$ are taken as given.

To obtain the time path of consumption in the continuous-time model, we at first obtain the intertemporal budget constraint.

Intertemporal Budget Constraint

In the continuous-time problem, the household's intertemporal budget constraint from time t is given by

$$\underbrace{a(t) + \int_t^\infty w(v) \exp\left(-\int_t^v r(s)ds\right) dv}_{\text{Lifetime income from time } t} = \underbrace{\int_t^\infty c(v) \exp\left(-\int_t^v r(s)ds\right) dv}_{\text{Lifetime consumption from time } t}. \tag{9}$$

The lifetime income at time t in turn consists of

1. $a(t)$: assets at time t;

2. $\int_t^\infty w(v) e^{-\int_t^v r(s)ds} dv$: the present value of wage income.
   $\rightarrow$ This is also called the human wealth, hereafter denoted by $h(t)$:

$$h(t) \equiv \int_t^\infty w(v) e^{-\int_t^v r(s)ds} dv.$$

Specification of u

In macroeconomics, the instantaneous utility u is often specified as the following:

$$u(c) = \begin{cases} \dfrac{c^{1-\theta} - 1}{1 - \theta} & \text{if } \theta \in \mathbb{R}_{++} \setminus \{1\}, \\ \\ \log c & \text{if } \theta = 1. \end{cases}$$

The above specification is called the CRRA utility function.
($*$) CRRA= Constant Relative Risk Aversion

Under the CRRA specification, $u'(c) = c^{-\theta}$ and $u''(c) = -\theta c^{-\theta-1}$.

Then, Euler equation is given by

$$\frac{\dot{c}_{t'}}{c_{t'}} = \frac{r_{t'} - \rho}{\theta}.$$

Consumption Function

From this Euler equation, we obtain

$$c_v = c(t) \exp\left(\frac{1}{\theta}\int_t^v (r(s) - \rho) ds\right). \tag{10}$$

Substituting (10) into the intertermporal budget constraint (9), we can obtain the consumption level at time t as follows:

$$c(t) = \eta(t)\left[a(t) + h(t)\right], \tag{11}$$

where $\eta(t)$ is the propensity to consume, defined by

$$\eta(t) \equiv \left[\int_t^\infty \exp\left(-\mathcal{R}(v)\right) dv\right]^{-1},$$
$$\mathcal{R}(v) \equiv \frac{1}{\theta}\int_t^v [(1-\theta)r(s) - \rho] ds.$$

Implication in a Simplified Case

When $\theta = 1$ (i.e., $u(c) = \log c$), $\eta(t) = \rho$, and then (11) becomes

$$c(t) = \rho(a(t) + h(t)).$$

Quiz

Show it.

$\downarrow$

Implication:

A household consumes a constant fraction of his/her "lifetime income," which includes the wage income at the future dates.

Such a consumption behavior is more realistic that in the Solow model, where he/she decides the level of current consumption based only on his/her "current income."

The Role of Expectation

Further simplify the model so that

1. $r(t)$ is constant at $r > 0$,
2. the household expects the wage will grow at a constant rate of $g > 0$,

where $r > g$ is assumed.

$\downarrow$

In this case, the consumption function (11) is further simplified to

$$c(t) = \rho \left(a(t) + \frac{w(t)}{r - g} \right). \tag{12}$$

We easily obtain the implication that if he/she expects the wage income grows faster (i.e., he expects $g \uparrow$), then, he/she immediately increases consumption at this moment.

Appendix

Proof of Theorem 1

- Let $\{c^*(t), a^*(t)\}$ denote the pair of time paths satisfying (1)–(3). Hereafter "(t)" is omitted unless to do so would cause confusions.
- Since u is strictly concave, we have the following inequality:

$$u(c^*) > u(c) + u'(c^*)(c^* - c),$$

where c can be arbitrarily chosen except c^*.

- Then, we have

$$\int_0^\infty e^{-\rho t} u(c^*) dt - \int_0^\infty e^{-\rho t} u(c) dt > D^*,$$

where

$$D^* \equiv \int_0^\infty e^{-\rho t} u'(c^*)(c^* - c) dt.$$

Therefore, we can show this theorem by showing $D^* \geq 0$.

Proof of Theorem 1

- Since both of $\{c^*(t)\}$ and $\{c(t)\}$ are feasible, D^* is rewritten as

$$D^* = \int_0^\infty e^{-\rho t} u'(c^*) \left[r(a^* - a) - (\dot{a}^* - \dot{a}) \right] dt.$$

- Using the method of integration by parts,

$$\begin{aligned}
\int_0^\infty e^{-\rho t} u'(c^*(t)) \dot{a}(t) dt &= \left[e^{-\rho t} u'(c^*(t)) a(t) \right]_{t=0}^\infty \\
&\quad - \int_0^\infty a(t) e^{-\rho t} \left(-\rho u'(c^*) + u''(c^*) \dot{c}^* \right) dt, \\
\int_0^\infty e^{-\rho t} u'(c^*(t)) \dot{a}^*(t) dt &= \left[e^{-\rho t} u'(c^*(t)) a^*(t) \right]_{t=0}^\infty \\
&\quad - \int_0^\infty a^*(t) e^{-\rho t} \left(-\rho u'(c^*) + u''(c^*) \dot{c}^* \right) dt.
\end{aligned}$$

- Substituting these results into D^* yields

Proof of Theorem 1

$$
\begin{aligned}
D^* = & \int_0^\infty e^{-\rho t} u'(c^*) \left[r - \rho + \frac{u''(c^*)}{u'(c^*)} \dot{c}^* \right] (a^* - a) dt \\
& - \lim_{t \to \infty} e^{-\rho t} u'(c^*(t)) a^*(t) + \lim_{t \to \infty} e^{-\rho t} u'(c^*(t)) a(t) \\
= & \lim_{t \to \infty} e^{-\rho t} u'(c^*(t)) a(t).
\end{aligned}
$$

- Finally, let $\lambda(t) = u'(c^*(t))$. Then, the Euler equation means $\dot{\lambda}(t)/\lambda(t) = \rho - r(t)$, which in turn implies

$$
\lambda(t) = \lambda(0) \exp\left(- \int_0^t r(s) ds \right).
$$

- Then, we obtain

$$
D^* = \lim_{t \to \infty} e^{-\rho t} u'(c^*(t)) a(t) = u'(c^*(0)) \lim_{t \to \infty} \left[a(t) \exp\left(- \int_0^t r(s) ds \right) \right],
$$

which is nonnegative from the NPG.