Review of the Solow Model
IEE.B402. Advanced Macroeconomics

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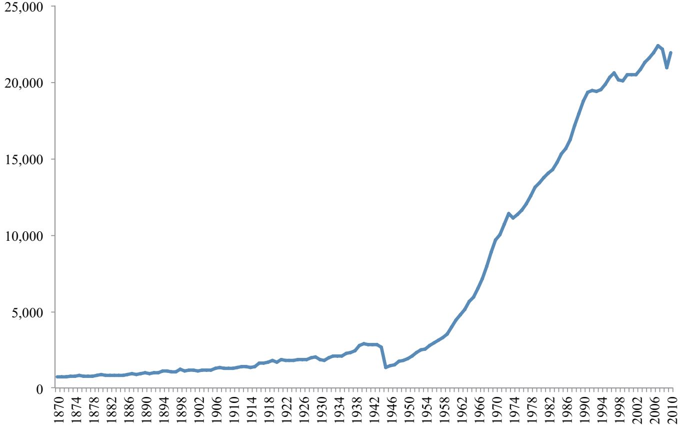
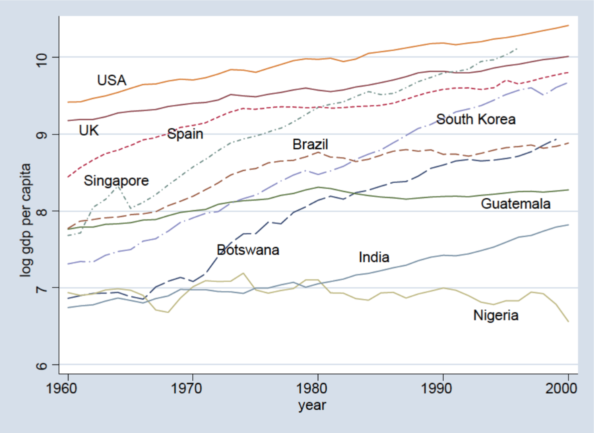
(Super) Long-run Trend in per-capita GDP in Japan

Source: Maddison Project (http://www.ggdc.net/maddison/), 1990US$



Trends in per-capita GDP in major countries (log scale)

References: Acemoglu (ch. 1, 2009), 2000 U$S



Introduction

- Before going to the models of modern macroeconomics, we at first briefly review the Solow growth model, which is covered also in the course of undergraduate macroeconomic theory.
- From the current point of view, the Solow model may have some shortcomings. Perhaps the most critical one is that this model lacks the agents' dynamically optimizing behaviors especially on the consumer side.
- Nevertheless, this model is useful as a starting point to think about the causes and the mechanics of economic growth by using a simple theoretical framework.

Introduction

- Except that the consumers do not intertemporally optimize their consumption-saving decisions, the Solow model has the following basic general equilibrium structure in common with the more advanced models explained later:
  - Households own the inputs (e.g., capital and labor) and assets of the economy,
  - Firms hire the inputs and use them to produce goods.
  - Markets exist.

  The following are the slides which are mainly based on ch.1 of Barro and Sala-i-Martin (2004) and ch. 2 of Acemoglu (2009).

Setup of the Solow Model

Basic Structure

- Closed economy.
- Time is continuous, denoted by $t \in [0, \infty)$.
- There is a single final good, used for consumption and investment.
    - We choose the final good as a numeraire, i.e., the price of this good is normalized to one.
- There are two primary factors, capital and labor.
- Economy is inhabited by a large number of households and firms
    - Households $=$ consumers, workers and capital owners.
    - Firms= producers.

      $(*)$ In order to grasp fundamental properties of the model, we ignore the government's activities here. So there is no tax or public spending.

Firms' Behavior: Production Function

- Assume all firms have access to the same production function.
  $\Rightarrow$ The economy admits a *representative firm*, with an aggregate production function.
- Throughout the slides, the aggregate production function for the final good is

$$Y(t) = F(K(t), L(t)), \tag{1}$$

  where
  - $Y(t)$ is the output of the final good at time t,
  - $K(t)$ is the input of capital stock at time t,
  - $L(t)$ is the input of labor at time t.
- Let us next impose the following standard assumptions on F.

Key Assumption of Production Function (1): Neoclassical Properties

Assumption 1

The production function F is twice differentiable in K and L, and satisfies

1. *Positive marginal products:*

$$F_1(K,L) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \;\; F_2(K,L) \equiv \frac{\partial F(\cdot)}{\partial L} > 0.$$

2. *Diminishing marginal products:*

$$F_{11}(K,L) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \;\; F_{22}(K,L) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0.$$

Key Assumption of Production Function (2): Linear Homogeneity

Assumption 2

The production function F is homogenous of degree one (or linearly homogenous):

$$F(\lambda K, \lambda L) = \lambda F(K, L) \ \forall \lambda \geq 0.$$

- Linearly homogenous production functions are particularly useful because of the following theorem.

Euler's theorem

Theorem 1

Suppose that the function $g(x,y)$ is differentiable in x and y, and that it is homogenous of degree m, i.e., $g(\lambda x, \lambda y) = \lambda^m g(x,y)$. Then,

1. $\dfrac{\partial g(x,y)}{\partial x}$ *and* $\dfrac{\partial g(x,y)}{\partial y}$ *are homogenous of degree* $m-1$.
2. $x\dfrac{\partial g(x,y)}{\partial x} + y\dfrac{\partial g(x,y)}{\partial x} = mg(x,y)$ *for all* x, y.

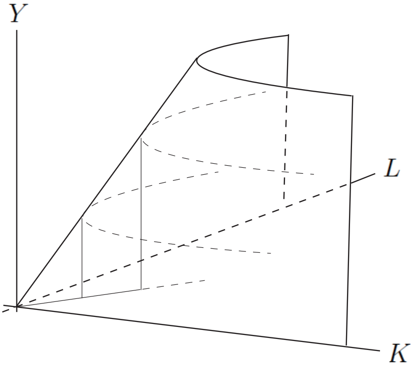
Key Assumption of Production Function (3): Inada Conditions

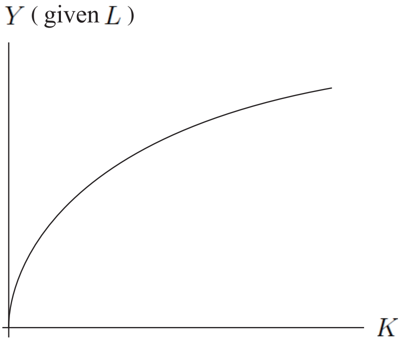
Assumption 3

The production function F satisfies the Inada conditions:

$$\lim_{K\to 0}\frac{\partial F(\cdot)}{\partial K}=\infty,\ \lim_{K\to\infty}\frac{\partial F(\cdot)}{\partial K}=0\ \forall L>0,$$
$$\lim_{L\to 0}\frac{\partial F(\cdot)}{\partial L}=\infty,\ \lim_{L\to\infty}\frac{\partial F(\cdot)}{\partial L}=0\ \forall K>0.$$

Typical Shape of Production Function with (1)–(3)





Quiz 1

Show that the following Cobb–Douglas production function satisfies Assumptions 1–3.

$$F(K, L) = AK^{\alpha}L^{1-\alpha}, \quad A > 0, \alpha \in (0, 1). \tag{2}$$

Firms' Behavior: Profit Optimization

- The representative firm chooses its inputs $K(t)$ and $L(t)$ so as to maximize its profit.
- Profit maximization problem is formulated as

$$\max_{K(t),L(t)} F(K(t),L(t)) - R(t)K(t) - w(t)L(t),$$

  where
  - $R(t)$ is the rental price of capital,
  - $w(t)$ is the wage rate.
- First-order conditions are

$$R(t) = F_K(K(t),L(t)), \tag{3}$$

$$w(t) = F_L(K(t),L(t)). \tag{4}$$

Quiz 2

- Suppose that there are J firms and the profit maximization problem of firm $j(= 1, 2, \ldots, J)$ is given by

$$\max_{K_j(t), L_j(t)} \; F(K_j(t), L_j(t)) - R(t)K_j(t) - w(t)L_j(t),$$

where K_j and L_j are demand for capital and labor of this firm.

- Let $K(t) = \sum_j K_j(t)$ and $L(t) = \sum_j L_j(t)$.

Show that also in this setup, $K(t)$ and $L(t)$ satisfy (3) and (4) from the profit maximization conditions and the assumption that F is linearly homogenous .

Firms' Behavior: Zero Profit Condition

Lemma 1

From (3), (4) and Assumption 2, it holds that

$$Y(t) = R(t)K(t) + w(t)L(t), \tag{5}$$

that is, firms earn no profits.

Proof.

Quiz 3 □

Households: Demographics and Labor Supply

- There is a continuum of households with measure $\overline{L}(t)$.
- Population grows at a constant rate of $n > 0$:

$$\dot{\overline{L}}(t)/\overline{L}(t) \left(\equiv \frac{d\overline{L}(t)/dt}{\overline{L}(t)} \right) = n, \tag{6}$$

which in turn means

$$\overline{L}(t) = \overline{L}(0) \exp(nt).$$

$(*)$ Hereafter, a dot over a variable " $\dot{}$ " indicates the time derivative.

- Each household is endowed with one unit of labor. Thus, $\overline{L}(t)$ also corresponds to the aggregate supply of labor.

Households: Budget Constraint

- The households also own the capital stock of the economy and rent it to firms. Let $\overline{K}(t)$ denote the aggregate supply of capital by the households.
- Therefore, the aggregate income flow of the households at time t is written as

$$\text{Income Flow} = R(t)\overline{K}(t) + w(t)\overline{L}(t).$$

- Accordingly, the aggregate budget constraint is expressed as

$$R(t)\overline{K}(t) + w(t)\overline{L}(t) = C(t) + S(t), \tag{7}$$

where

  - $C(t)$ is the aggregate amount of consumption,
  - $S(t)$ is that of savings.

Households: Note

- In most textbooks of macroeconomics, the market-clearing conditions for labor and capital are imposed in advance:

$$L(t) = \overline{L}(t), \quad K(t) = \overline{K}(t) \text{ for all } t \in [0, \infty).$$

- Then, hereafter we use $L(t)$ and $K(t)$, rather than $\overline{L}(t)$ and $\overline{K}(t)$, unless to do so would cause confusions.
- From Proposition 1 and the budget constraint (7), the aggregate income flow is then rewritten as

$$\text{Income Flow} = R(t)K(t) + w(t)L(t) = Y(t). \tag{8}$$

Households: Income Flow and Saving Behavior

The following is the assumption made by Solow (1956), and Swan (1956):

Assumption 4 (Solow (1956) and Swan (1956))

The households save a constant fraction $s \in (0, 1)$ *of their income: that is,*

$$S(t) = sY(t), \ C(t) = (1 - s)Y(t). \tag{9}$$

- That is, there is a lack of optimization on the household side. Although such a exogenous saving rate is a convenient starting point, this assumption is restricted.
- This is why we need a method of dynamic optimization in modern macroeconomics, explained later.

Equilibrium of the Solow Model

Market-clearing Conditions

- The market-clearing conditions for production factors ($K = \overline{K}$ and $L = \overline{L}$) have been already considered.
- Since we consider the closed-economy, the market-clearing conditions for the final good and the financial resources are respectively given by

$$Y(t) = C(t) + I(t), \tag{10}$$

$$S(t) = I(t), \tag{11}$$

where $I(t)$ is the aggregate investment.

Lemma 2

If (10) holds, then (11) is automatically satisfied, and vice versa.

Note: Walras' Law

- More generally, from (5) and (7), we can obtain the following equations:

$$R(t)\left[K(t)-\overline{K}(t)\right]+w(t)\left[L(t)-\overline{L}(t)\right]+\left[C(t)+I(t)-Y(t)\right]$$
$$=I(t)-S(t),$$

  where
  - the left-hand-side is the sum of excess demand for the goods and factors,
  - the right-hand-side is the excess demand for financial resources.
- The above equation means that if the three conditions are satisfied from the four market-clearing conditions, the rest automatically implies. This is called the Walras' law.

Equilibrium Dynamics

- From time t to $t + \Delta t$, the capital stock evolves according to

$$K(t+\Delta t) - K(t) = \underbrace{I(t)\Delta t}_{\text{Investment}} - \underbrace{\delta K(t)\Delta t}_{\text{Depreciation}},$$

  where $\delta > 0$ is the depreciation rate.

- Taking a limit of $\Delta t \to 0$,

$$\dot{K}(t) = I(t) - \delta K(t). \tag{12}$$

Definition of Equilibrium

Definition 1

Given a path of population $L(t)$ *in (6) and the initial capital stock* $K(0) > 0$*, an equilibrium of the Solow model is characterized by the paths of*

- *output,* $Y(t)$ *satisfying (1),*
- *capital stock,* $K(t)$ *satisfying (12),*
- *factor prices,* $R(t)$ *and* $w(t)$ *satisfying (3) and (4),*
- *savings and investment,* $S(t)$ *and* $I(t)$ *satisfying (9) and (11)*

The Fundamental Equation of the Solow Model

- Using (1), (9) and (11), the dynamics of capital stock (12) is rewritten as

$$\dot{K}(t) = sF(K(t), L(t)) - \delta K(t). \tag{13}$$

  Thus, given the initial conditions $K(0)$ and $L(0)$, (6) and (13) determines the path $[K(t), L(t)]_{t=0}^{\infty}$.
- To obtain the equilibrium more clearly, it is useful to introduce the capital–labor ratio of the economy:

$$k(t) \equiv K(t)/L(t),$$

  which implies that

$$\dot{k}(t)/k(t) = \dot{K}(t)/K(t) - n \to \dot{k}(t) = \dot{K}(t)/L(t) - nk(t). \tag{14}$$

The Fundamental Equation of the Solow Model

- Substituting (13) into (15) and using Assumption 2, we have the dynamics of $k(t)$ as

$$\dot{k}(t) = sf(k(t)) - (n + \delta)k(t). \tag{15}$$

where the function f is defined as $f(k) \equiv F(k, 1)$.

- Therefore, the equilibrium of the Solow model is defined in an easier way.

Definition 2

Given the initial condition $k(0) > 0$, an equilibrium of the Solow model is the path of capital-labor ratio $[k(t)]_{t=0}^{\infty}$ satisfying (15).

Quiz 5

Show that from (3), (4) and the definition of f, it holds that

$$R(t) = f'(k(t)),$$
$$w(t) = f(k(t)) - k(t)f'(k(t)),$$

where $f'(\cdot) \equiv df(\cdot)/dk$.

Steady State and Transitional Dynamics

- Once the path of capital-labor ratio $[k(t)]_{t=0}^{\infty}$ is determined, the paths of other variables are accordingly determined: for example,

$$\begin{aligned} Y(t) &= f(k(t))L(t), \\ S(t)(= I(t)) &= sY(t), \\ C(t) &= (1-s)Y(t), \\ R(t) &= f'(k(t)), \\ w(t) &= f(k(t)) - k(t)f'(k(t)), \end{aligned}$$

- Thus, it is very important to understand the dynamic properties of $[k(t)]_{t=0}^{\infty}$.

Steady State and Transitional Dynamics

- At first, we can easily show

Lemma 3

Under the Assumptions 1 and 3,

$$f''(k) < 0, \ \lim_{k \to 0} f'(k) = \infty, \ \lim_{k \to \infty} f'(k) = 0.$$

Proof.

Quiz 6 □

- Then, we arrive at the following proposition.

Steady State and Transitional Dynamics

Proposition 1

Suppose that Assumptions 1–3 hold. Then, in the Solow model,

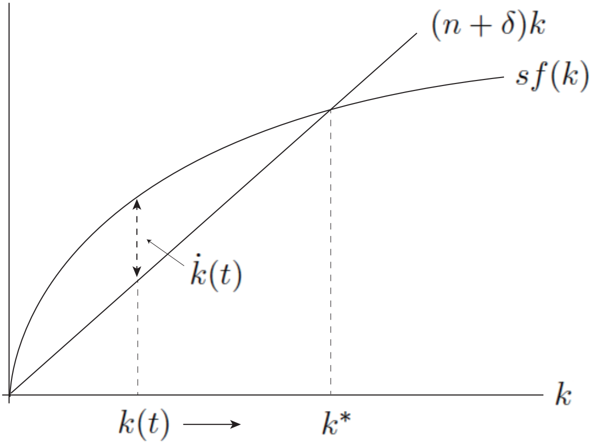
1. *There is a unique steady state* k^* *such that*

$$sf(k^*) = (n + \delta)k^*, \tag{16}$$

   namely, $k(t)$ *becomes constant at* $k(t) = k^*$*.*

2. *Starting from any* $k(0) > 0$, $k(t)$ *monotonically converges to* k^**.*

Graphical Analysis



An Example: Cobb–Douglas Technology

- We can illustrate the results for the case of a Cobb–Douglas production function (2).
- The dynamics of the capital-labor ratio is

$$\dot{k}(t) = sAk(t)^{\alpha} - (n+\delta)k(t).$$

- The steady-state level of k and output per capita (denoted by $y \equiv Y/L$) are given by

$$k^{*} = [sA/(n+\delta)]^{1/(1-\alpha)}, \tag{17}$$

$$y^{*} = Ak^{*\alpha} = A^{1/(1-\alpha)}[s/(n+\delta)]^{\alpha/(1-\alpha)}. \tag{18}$$

A Closed-form Solution under Cobb-Douglas Technology

- When the production function is Cobb-Douglas and the saving rate is constant, it is possible to get a closed-form solution for $k(t)$:

$$k(t) = v(t)^{1/(1-\alpha)},$$

where $v(t)$ is

$$v(t) = \frac{sA}{n+\delta} + \left[k(0)^{1-\alpha} - \frac{sA}{n+\delta}\right] \exp[-(1-\alpha)(n+\delta)t].$$

In this case, we can explicitly obtain the result that $k(t) \to k^*$ as $t \to \infty$.

Taking Stock

- The Solow model provides as a simple and tractable framework of economic growth.
- If there is no technological progress, and the production function satisfies the neoclassical properties (i.e., Assumption 1), there will be no sustained growth.

References

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- Barro, R. J. and X. Sala-i-Martin (2004) *Economic Growth*, Cambridge MIT Press.
- Solow, R. M. (1956) "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics* 70, 65–94.
- Swan T. W. (1956) "Economic Growth and Capital Accumulation," *Economic Record* 32, 334–361.