A Discrete-Time Ramsey–Cass–Koopmans Model

IEE.B402. Advanced Macroeconomics

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Plan

A discrete-time Ramsey–Cass–Koopmans model

- Setup
- Households' and firms' behavior
- Closing the model
- Characterization of the competitive equilibrium path

An extension: introducing the households' labor-leisure choice

- Setup
- Households' and firms' behavior
- Closing the model

Setup

- Time is discrete and extends from zero to infinity: $t = 0, 1, 2, \dots$
- Two types of economic agents: households and firms
- Population of households is fixed at $\bar{L} > 0$, that is, there is no population growth
- There is a single final good used for consumption and investment
- The final good is produced from capital and labor
- Perfect competition

($*$) Notations are basically the same as the continuous-time Ramsey model. (e.g., $c_t \in \mathbb{R}_+$ is consumption, $a_t \in \mathbb{R}$ is asset ...)

Households' Behavior: Utility Maximization Problem

- Households are homogeneous
- A representative household's dynamic utility maximization problem:

$$\max_{\{c_t, a_{t+1}\}} \quad U = \sum_{t=0}^{\infty} \beta^t u(c_t), \quad \beta \in (0,1)$$

$$\text{s.t.} \quad a_{t+1} = (1 + r_t)a_t + w_t - c_t \quad t = 0, 1, 2, \ldots, \qquad (1)$$

$$\lim_{T\to\infty} \frac{a_{T+1}}{\prod_{j=1}^{T}(1 + r_j)} \geq 0$$

  where β is the discount factor
- Assumptions:

  $u'(c) > 0 \; \forall c \in \mathbb{R}_+$

  $u''(c) < 0 \; \forall c \in \mathbb{R}_+$

  $\lim_{c\to 0^+} u'(c) = \infty$

Households' Behavior: Euler equation and TVC

- Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Big\{ u(c_t) + \lambda_t \big[(1 + r_t) a_t + w_t - c_t - a_{t+1} \big] \Big\}$$

  where λ_t is the Lagrangian multiplier.

- First-order-conditions (F.O.Cs):

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} = 0 \forall t : &\quad u'(c_t) = \lambda_t \\ \frac{\partial \mathcal{L}}{\partial a_{t+1}} = 0 \forall t : &\quad \lambda_t = \beta(1 + r_{t+1}) \lambda_{t+1} \end{aligned}$$

  From these equations, we obtain

$$u'(c_t) = \beta u'(c_{t+1})(1 + r_{t+1}) \tag{2}$$

Households' Behavior: Euler equation and TVC

- The transversality condition (TVC) is given by

$$\lim_{T\to\infty} \beta^T u'(c_T) a_{T+1} = 0 \tag{3}$$

- Using (2), we can express (3) also as

$$\lim_{T\to\infty} \frac{a_{T+1}}{\prod_{j=1}^{T}(1+r_j)} = 0$$

Firms' Behavior: Production Function

Production function: $Y_t = F(K_t, L_t)$

Assumption 1

The production function F is twice differentiable in K and L, and satisfies

1. *Positive marginal products:*

$$F_1(K,L) \equiv \frac{\partial F(\cdot)}{\partial K} > 0,\ F_2(K,L) \equiv \frac{\partial F(\cdot)}{\partial L} > 0$$

2. *Diminishing marginal products:*

$$F_{11}(K,L) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0,\ F_{22}(K,L) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0$$

3. *Homogeneous of degree one (or linearly homogenous):*

$$F(zK, zL) = \lambda F(K,L)\ \forall z \geq 0$$

Firms' Behavior: Profit Maximization

- A representative firm's profit maximization problem:

$$\max_{K_t, L_t} F(K_t, L_t) - R_t K_t - w_t L_t$$

F.O.Cs:

$$R_t = F_1(K_t, L_t), \quad w_t = F_2(K_t, L_t)$$

- Since F is homogenous of degree 1, it holds true that

$$F(K_t, L_t) = f(K_t/L_t)L_t$$

where $f(x) \equiv F(x, 1)$

- Then, the above F.O.Cs are rewritten as:

$$R_t = f'(K_t/L_t) \quad (4)$$

$$w_t = f(K_t/L_t) - \frac{K_t}{L_t} f'(K_t/L_t) \quad (5)$$

where $f'(K/L) = df(K/L)/d(K/L)$.

Closing the Model

- Market-clearing condition for assets is given by

$$a_t \bar{L} = K_t \tag{6}$$

Let $k_t \equiv K_t/\bar{L}$ denote physical capital in per capita terms.
$\rightarrow$(6) implies $a_t = k_t$.

- The interest rate r_t and the rental price of capital R_t satisfy

$$R_t - \delta = r_t \tag{7}$$

- Market-clearing condition for labor:

$$\bar{L} = L_t \tag{8}$$

Autonomous Dynamic System

- Using (1)–(8), we obtain

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t \tag{9}$$

$$u'(c_t) = \beta u'(c_{t+1})(f'(k_{t+1}) + 1 - \delta) \tag{10}$$

$$\lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0 \tag{11}$$

Quiz

Show it.

Definition 1

Given $k_0 > 0$, the pair of sequences $\{k_t, c_t\}$ satisfying (9)–(11) is the competitive equilibrium path.

The procedure to characterize the equilibrium is the same as the continuous-time model.

Steady State

- As is already explained, the steady state is the situation where the variables given by the dynamic system become constant over time.

- Let (k^*, c^*) denote the steady state of the system (9)–(11).
- From (10) with $c_t = c_{t+1}$,

$$1 = \beta(f'(k^*) + 1 - \delta) \Leftrightarrow f'(k^*) = 1/\beta - (1 - \delta) > 0 \qquad (12)$$

- As already shown in the slides on the Solow model, f satisfies the following three properties:

$$f''(k) < 0, \quad \lim_{k \to 0} f'(k) = \infty, \quad \lim_{k \to \infty} f'(k) = 0$$

$\rightarrow$This means that there uniquely exists $k^* > 0$ that satisfies (12)

Steady State

- Once we have obtained k^*, we have

$$
\begin{aligned}
c^* &= f(k^*) + (1-\delta)k^* - k^* \\
&= f(k^*) - \delta k^*
\end{aligned}
$$

($*$) If we specify u and f respectively as $u(c) = \ln c$ and $f(k) = Ak^\alpha$ with $A > 0$ and $\alpha \in (0,1)$, we can explicitly derive k^* and c^* as follows:

$$
\begin{aligned}
k^* &= \left[\frac{\alpha A}{1/\beta - (1-\delta)}\right]^{1/(1-\alpha)} > 0 \\
c^* &= Ak^{*\alpha} - \delta k^* \\
&= \left(\frac{\alpha A}{1/\beta - (1-\delta)}\right)^{1/(1-\alpha)} \frac{1/\beta - 1 + (1-\alpha)\delta}{\alpha} > 0
\end{aligned}
$$

Linear Approximation of the System

- Linear approximation of (9) around the steady state (k^*, c^*):

$$\begin{aligned} k_{t+1} &= k^* + \left[f'(k^*) + (1-\delta)\right](k_t - k^*) - (c_t - c^*) \\ &= k^* + (1/\beta)(k_t - k^*) - (c_t - c^*) \end{aligned} \tag{13}$$

- Linear approximation of (10) around the steady state (k^*, c^*):

$$\begin{aligned} u'(c^*)(c_t - c^*) &= \beta u'(c^*) f''(k^*)(k_{t+1} - k^*) + u''(c^*)(c_{t+1} - c^*) \\ c_{t+1} - c^* &= -\frac{\beta u'(c^*) f''(k^*)}{u''(c^*)}(k_{t+1} - k^*) + (c_t - c^*) \end{aligned}$$

Substituting (13) into the above equation yields

$$c_{t+1} - c^* = -\frac{u'(c^*) f''(k^*)}{u''(c^*)}(k_t - k^*) + \left(1 + \frac{\beta u'(c^*) f''(k^*)}{u''(c^*)}\right)(c_t - c^*) \tag{14}$$

Local Stability of Linearized System

- Let $q^* = \dfrac{u'(c^*)f''(k^*)}{u''(c^*)} > 0$

- Then, the linearized system around the steady state (k^*, c^*) is given by

$$\begin{pmatrix} k_{t+1} - k^* \\ c_{t+1} - c^* \end{pmatrix} = J \begin{pmatrix} k_t - k^* \\ c_t - c^* \end{pmatrix}$$

where J is the Jacobian matrix:

$$\begin{pmatrix} 1/\beta & -1 \\ \\ -q^* & 1 + \beta q^* \end{pmatrix}$$

Local Stability of Linearized System

- Characteristic equation:

$$p(\omega) \equiv \omega^2 - (\mathrm{tr}J)\omega + \det J = 0 \tag{15}$$

Note that

$$p(0) = \det J = 1/\beta > 0$$
$$p(1) = 1 - \mathrm{tr}J + \det J = -\beta q^* < 0$$

Then, we can find

1. Both roots are positive, and
2. One root is strictly less than one, while the other is strictly greater than one.

That is, (k^*, c^*) is a saddle.

Local Stability of Linearized System

- Then, the general solution is

$$\begin{pmatrix} k_t - k^* \\ c_t - c^* \end{pmatrix} = z_1 \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} (\omega_1)^t + z_2 \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} (\omega_2)^t \qquad (16)$$

  where

  1. ω_j $(j = 1, 2)$ is the eigenvalue of matrix J.
     $\Rightarrow$ Without any loss of generality, let $\omega_1 > 1$ and $0 < \omega_2 < 1$.
  2. $v_j \equiv (v_{1j}, v_{2j})^T$ is the eigenvector corresponding to the eigenvalue ω_j.
  3. z_j $(j \in \{1, 2\})$ is a constant value *still to be determined*.

- Imposing $t = 0$ in (16):

$$\begin{pmatrix} k_0 - k^* \\ c_0 - c^* \end{pmatrix} = z_1 \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} + z_2 \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$$

Determination of c_0

- Because the initial value of physical capital in per capita, k_0 is exogenously given, z_1 and z_2 must satisfy

$$z_1 = 0, z_2 = \frac{k_0 - k^*}{v_{12}}$$

- Thus, the initial consumption, c_0, must be determined such that $c_0 = c^* + \dfrac{v_{22}}{v_{12}}(k_0 - k^*)$.

- Therefore, from Eq. (16), we can analytically obtain the optimal growth path as follows:

$$k_t - k^* = (k(0) - k^*)(\omega_2)^t, \tag{17}$$

$$\begin{aligned} c_t - c^* &= (v_{21}/v_{11})(k_t - k^*) \\ &= (v_{21}/v_{11})(k_0 - k^*)(\omega_2)^t \end{aligned} \tag{18}$$

An extension: introducing the households' labor-leisure choice

Setup

- The structure is basically same as the baseline model.
- Each household is endowed with one unit of time. Now we suppose that he/she divides the time between
  (i) hours worked, denoted by $h_t \in [0, 1]$, and
  (ii) leisure, denoted by $\ell_t \in [0, 1]$,
  where $h_t + \ell_t = 1$.
- The representative household's lifetime utility:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t)$$

- Moreover, we specify the one-period utility function u as

$$\begin{aligned} u(c_t, \ell_t) &= \ln c_t + \zeta \ln \ell_t \\ &= \ln c_t + \zeta \ln(1 - h_t) \end{aligned} \tag{19}$$

Households' Behavior

- Utility maximization problem:

$$\max_{\{c_t, h_t, a_{t+1}\}} \quad U = \sum_{t=0}^{\infty} \beta^t \left[\ln c_t + \zeta \ln(1 - h_t)\right]$$

$$\text{s.t.} \quad a_{t+1} = (1 + r_t)a_t + w_t h_t - c_t \quad t = 0, 1, 2, \ldots \qquad (20)$$

$$\lim_{T\to\infty} \frac{a_{T+1}}{\prod_{j=1}^{T}(1 + r_j)} \geq 0$$

- Conditions for utility maximization:

$$c_{t+1}/c_t = \beta(1 + r_{t+1}) \qquad (21)$$

$$\frac{\zeta c_t}{1 - h_t} = w_t \qquad (22)$$

and the TVC (3)

Firms' Behavior

- Firms' behavior is essentially the same, that is, the conditions for profit maximization are given by (4) and (5).
- Hereafter, we specify the function F as $F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha}$, or equivalently, $f(K_t/L_t) = A(K_t/L_t)^{\alpha}$, where $A > 0$ and $\alpha \in (0, 1)$.
- (4) and (5) are accordingly reduced to

$$R_t = \alpha A \left(\frac{K_t}{L_t} \right)^{\alpha - 1} \tag{4'}$$

$$w_t = (1 - \alpha) A \left(\frac{K_t}{L_t} \right)^{\alpha} \tag{5'}$$

Equilibirum

- The market-clearing condition for assets is given by (6)
- As in the baseline model, we let $k_t \equiv K_t/\bar{L}$ denote physical capital in per capita terms $\rightarrow$ (6) implies $a_t = k_t$.
- On the other hand, the market-clearing condition for labor (8) is now replaced by

$$\bar{L}h_t = L_t \tag{23}$$

- From (4) and (5), we obtain

$$R_t = \alpha A \left(\frac{k_t}{h_t}\right)^{\alpha-1} \tag{24}$$

$$w_t = (1-\alpha)A\left(\frac{k_t}{h_t}\right)^{\alpha} \tag{25}$$

Autonomous Dynamic System

- From (6), (7), and (20)–(25), we obtain

$$k_{t+1} = Ak_t^{\alpha} h_t^{1-\alpha} + (1-\delta)k_t - c_t \tag{26}$$

$$c_{t+1} = c_t \beta \left[\alpha A \left(\frac{k_{t+1}}{h_{t+1}} \right)^{-(1-\alpha)} + 1 - \delta \right] \tag{27}$$

$$c_t = \frac{1-\alpha}{\zeta} A \left(\frac{k_t}{h_t} \right)^{\alpha} (1-h_t) \tag{28}$$

Quiz

Show it.

The TVC is given by (11) in the equilibrium.

Definition 2

Given $k_0 > 0$, the triple of the sequences $\{k_t, c_t, h_t\}$ satisfying (26)-(28) together with (11) is the competitive equilibrium path.