

Ramsey–Cass–Koopmans Model (1)

IEE.B402. Advanced Macroeconomics

Ryoji Ohdoi

Department of Industrial Engineering and Economics,
Tokyo Institute of Technology

Plan

- ▶ Setup of the model
 - ▶ Households and firms
 - ▶ Closing the model
- ▶ Competitive equilibrium path
 - ▶ Definition
 - ▶ Steady state
 - ▶ Transitional dynamics
- ▶ The social planner's problem

Introduction

- ▶ Recall that in the Solow-Swan model, consumption C and savings S are proportional to current income Y :

$$S = sY, \quad C = (1 - s)Y,$$

where $s \in (0, 1)$ is the saving rate which is assumed to be exogenous.

- ▶ Ramsey-Cass-Koopmans model (or simply, Ramsey model):
 - ▶ This model differs from the Solow-Swan model in the respect that it endogenizes the savings rate by explicitly modeling the consumer's infinite-horizon dynamic optimization.
- ▶ Although the original Ramsey-Cass-Koopmans model is the model in continuous time, I will initially develop this model in discrete time.

Model

Households: Demographics and Utility Function

Population growth:

$$\dot{L}(t)(\equiv dL(t)/dt) = nL(t) \Leftrightarrow L(t) = L(0)e^{nt}. \quad (1)$$

(*) $L(0)$ is normalized to one.

Lifetime utility function:

$$\begin{aligned} U &= \int_0^{\infty} e^{-\rho t} L(t) u(c(t)) dt \\ &= \int_0^{\infty} e^{-(\rho-n)t} u(c(t)) dt \end{aligned} \quad (2)$$

Assumption 1

$$\rho > n$$

Households

Let $\mathcal{A}(t)$ denote asset holdings of the representative household at t .

Flow budget constraint:

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - c(t)L(t) \quad (3)$$

► $r(t)$: interest rate.

(*) We will discuss the relationship between r and R soon later.

Define a as follows:

$$a(t) \equiv \frac{A(t)}{L(t)}$$

Then, we obtain the flow budget equation in per-capita terms:

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t) \quad (4)$$

Utility Maximization Problem

The representative household's utility maximization problem

$$\begin{aligned} \max_{\{a(t), c(t)\}} \quad & U(0) = \int_0^{\infty} \exp(-(\rho - n)t) u(c(t)) dt \\ \text{s.t.} \quad & \dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t) \\ & \lim_{t \rightarrow \infty} a(t) \exp\left(-\int_0^t (r(s) - n) ds\right) \geq 0 \end{aligned}$$

with taking the following variables as given:

1. $a(0)$: the initial condition
2. Times paths of $r(t)$ and $w(t)$, which implies that each household is a price taker

Euler Equation and Transversality Condition

Euler equation:

$$\sigma(c_t) \frac{\dot{c}(t)}{c(t)} = r(t) - \rho. \quad (8)$$

► $\sigma(c) \equiv -\frac{cu''(c)}{u'(c)} > 0 \forall c$

TVC:

$$\lim_{t \rightarrow \infty} a(t) \exp \left(- \int_0^t (r(s) - n) ds \right) = 0 \quad (\because \nu > 0). \quad (9)$$

No-Arbitrage Condition

Each household has the opportunity to hold the following two types of assets

1. capital $k(t)$
2. individual bonds $b_p(t)$

(*) When introduce the government's activity, public bonds are added.

The “no-arbitrage-connection” btw the two assets:

$$R(t) - \delta = r(t) \tag{10}$$

Firms

A representative firm's profit maximization problem:

$$\max_{K(t), L(t)} F(K(t), L(t)) - R(t)K(t) - w(t)L(t),$$

Since $F(K, L) = f(k)L$, the above problem can be converted to

$$\max_{k(t), L(t)} [f(k(t)) - R(t)k(t) - w(t)]L(t)$$

Competitive factor markets then imply:

$$R(t) = f'(k(t)), \quad (11)$$

and

$$w(t) = f(k(t)) - k(t)f'(k(t)). \quad (12)$$

Market-Clearing Conditions

- ▶ Let $B_p(t)$ denote the aggregate amount of individual bonds ($A(t) = K(t) + B_p(t)$)
- ▶ Since the “lending&borrowing” in an individual sense is cancelled out as a whole,

$$B_p(t) = 0.$$

- ▶ Then, asset market-clearing condition in *per capita terms* is given by

$$a(t) = k(t). \quad (14)$$

Competitive equilibrium path

Definition

- ▶ The zero-profit-condition of firms implies

$$R(t)k(t) + w(t) = f(k(t))$$

↓

- ▶ The household's budget constraint is rewritten as

$$\begin{aligned}\dot{k}(t) &= (R(t) - \delta - n)k(t) + w(t) - c(t) \\ &= f(k(t)) - (n + \delta)k(t) - c(t)\end{aligned}\tag{16}$$

which is equivalent to the market-clearing condition for the final good.

Definition

- Since $r(t) = R(t) - \delta = f'(k(t)) - \delta$, the Euler eq. becomes

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma(c(t))} (f'(k(t)) - \delta - \rho) \quad (17)$$

- TVC is now expressed as

$$\lim_{t \rightarrow \infty} k(t) \exp \left(- \int_0^t [f'(k(s)) - (n + \delta)] ds \right) = 0 \quad (18)$$

Definition

Definition 1 (Competitive Equilibrium Path)

Given $k(0) > 0$, the pair of paths $\{k(t), c(t)\}$ which jointly satisfy (16)–(18) constitute a competitive equilibrium path.

► Eqs. (16) – (18) :

→ The system of differential equations wrt $k(t)$ and $c(t)$ with the terminal condition

→ These equations are called the **Autonomous Dynamic System**

How can we derive the competitive equilibrium path?

Basically we use the following procedure:

1. Examine the existence and uniqueness of the **Steady State**
 - ▶ **SS**= the path in which $k(t)$ and $c(t)$ are constant.
2. Examine the stability of the steady state:
3. Check whether or not the **Transitional Dynamics** to the steady state is uniquely determined or not.

Existence and Uniqueness of Steady State

Steady State

- From the Euler equation with $\dot{c} = 0$,

$$\frac{\dot{c}(t)}{c(t)} = 0 \Leftrightarrow \frac{1}{\sigma(c(t))} (f'(k(t)) - \delta - \rho) = 0.$$

- Then, k^* is determined as

$$f'(k^*) = \rho + \delta > n + \delta. \quad (19)$$

(*) Eq. (19) pins down the steady-state capital-labor ratio only as a function of the production function, the discount rate and the depreciation rate

Steady State

- Once, k^* is determined,

$$c^* = C(k^*) \equiv f(k^*) - (n + \delta)k^*. \quad (20)$$

- Note that

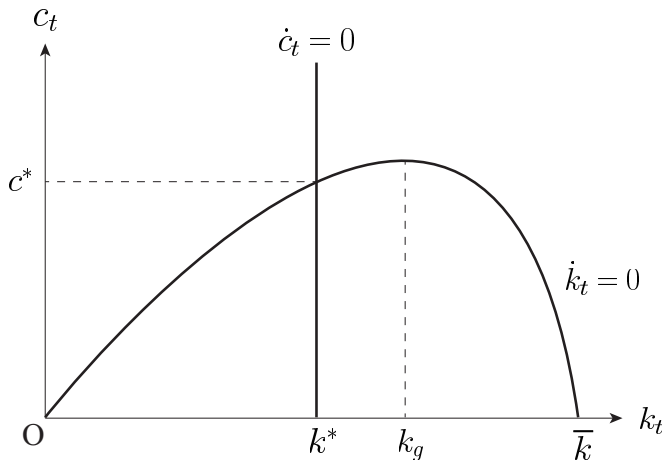
$$C'(k) = f'(k) - (n + \delta) \gtrless 0 \Leftrightarrow f'(k) \gtrless n + \delta.$$

- Let us define k_g such that $f'(k_g) = n + \delta$.
 k_g is golden rule of capital stock.

- Since $f'' < 0$,

$$\frac{dc^*}{dk^*} \gtrless 0 \Leftrightarrow k \lesseqgtr k_g.$$

Steady State



Modified Golden Rule

- ▶ k^* is called the **modified golden rule**, which is smaller than the golden rule.
- ▶ Note that k^*
 - ▶ maximizes life-time utility, because it is derived by solving the utility maximization problem,
 - ▶ does NOT maximize steady-state consumption (to see why, see figure)

This implies that achieving the golden rule is not desirable from the viewpoint of utility maximizing.

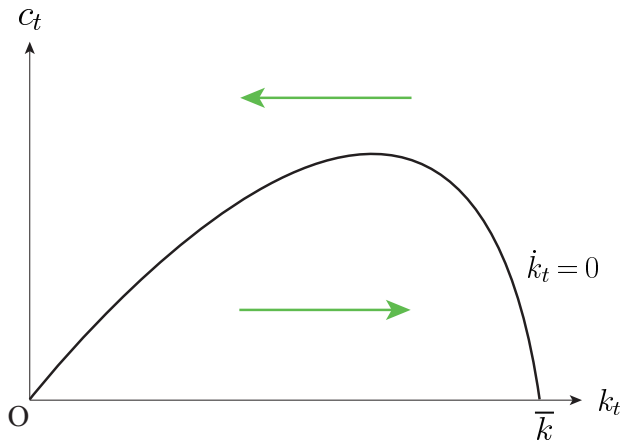
Stability of SS and transitional dynamics

Analysis (1): Graphical analysis using **Phase Diagram**

Transitional Dynamics

From the dynamics of $k(t)$,

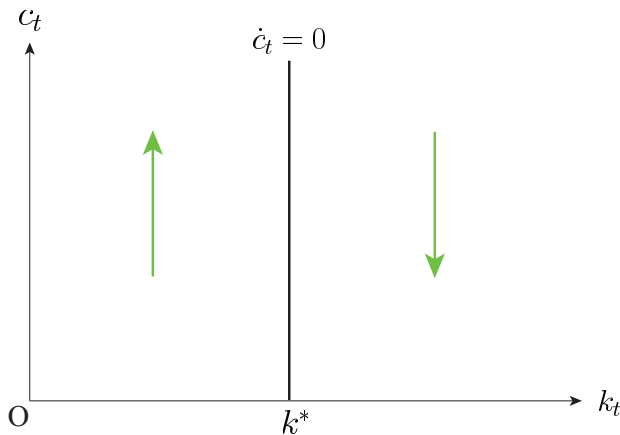
$$\dot{k}(t) \gtrless 0 \Leftrightarrow c(t) \lesseqgtr f(k(t)) - (n + \delta)k(t) \equiv C(k(t)).$$



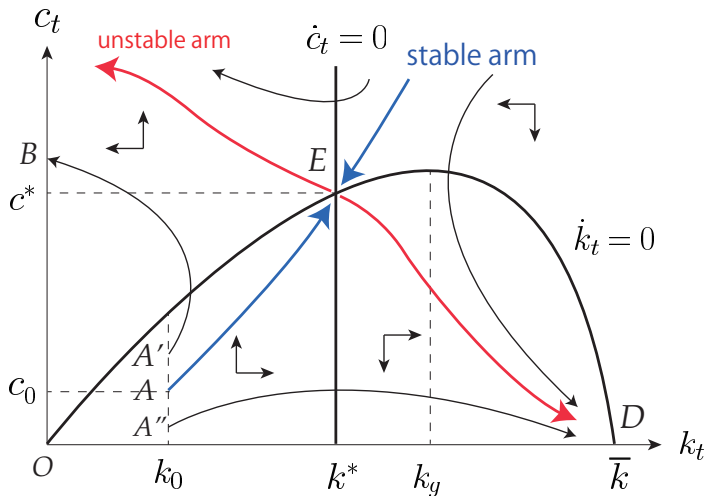
Transitional Dynamics

From the dynamics of $c(t)$ with $c(t) > 0$,

$$\dot{c}(t) \gtrless 0 \Leftrightarrow k(t) \lesseqgtr k^*.$$



Transitional Dynamics



Stability of SS and transitional dynamics

Analysis (2): Analysis using **Linear Approximation of System (16)–(17)**

Local Stability of Linearized System

- ▶ Linear approximation of (16) in the neighborhood of SS:

$$\begin{aligned}\dot{k}(t) &= [f'(k^*) - (n + \delta)](k(t) - k^*) - (c(t) - c^*) \\ &= (\rho - n)(k(t) - k^*) - (c(t) - c^*)\end{aligned}$$

- ▶ Linear approximation of (17) in the neighborhood of SS:

$$\begin{aligned}\dot{c}(t) &= \frac{c^* f''(k^*)}{\sigma(c^*)} (k(t) - k^*) \\ &= -\frac{u'(c^*) f''(k^*)}{u''(c^*)} (k(t) - k^*)\end{aligned}$$

Local Stability of Linearized System

∴ The linearized (or local) dynamics:

$$\begin{pmatrix} \dot{k}(t) \\ \dot{c}(t) \end{pmatrix} = J \begin{pmatrix} k(t) - k^* \\ c(t) - c^* \end{pmatrix}$$

where J is Jacobian matrix:

$$J = \begin{pmatrix} \rho - n & -1 \\ -\frac{f''(k^*)u'(c^*)}{u''(c^*)} & 0 \end{pmatrix}$$

Local Stability of Linearized System

It is well known that the general solution is

$$\begin{pmatrix} k(t) - k^* \\ c(t) - c^* \end{pmatrix} = Z_1 \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} \exp(\omega_1 t) + Z_2 \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} \exp(\omega_2 t) \quad (24)$$

- ▶ ω_j ($= 1, 2$): the eigenvalue of matrix J ;
- ▶ $v_j \equiv (v_{1j}, v_{2j})^T$: the eigenvector corresponding to ω_j ($j \in \{1, 2\}$);
- ▶ Z_j ($j \in \{1, 2\}$): a constant value *still to be determined*

Characteristic Equation

- ▶ ω_j is determined from the following *characteristic equation*:

$$\det(J - \omega I) = 0 \Leftrightarrow \det \begin{pmatrix} \rho - n - \omega & -1 \\ \frac{f''(k^*)c^*}{\varepsilon_u(c^*)} & 0 - \omega \end{pmatrix} = 0,$$
$$\Leftrightarrow \omega^2 - (\rho - n)\omega + \frac{f''(k^*)c^*}{\varepsilon_u(c^*)} = 0.$$

- ▶ It is shown that there are two real eigenvalues, one negative and one positive (consider the reason why).
- ▶ Without any loss of generality, let $\omega_1 > 0$ and $\omega_2 < 0$ respectively denote the positive and the negative eigenvalues.

Determination of Initial Consumption

Determination of $c(0)$:

$$\begin{aligned}k(0) - k^* &= Z_1 v_{11} + Z_2 v_{12} \text{ and} \\c(0) - c^* &= Z_1 v_{21} + Z_2 v_{22}\end{aligned}\tag{25}$$

Thus, the initial consumption, $c(0)$, is determined such that

1. $Z_1 = 0$: otherwise the economy diverges from the steady state, and such a path violates either the Keynes-Ramsey rule or the TVC;
2. $Z_2 = (k(0) - k^*)/v_{12}$: otherwise (25) does not hold given $k(0)$.

Lemma 2

The initial consumption is determined as $c(0) = c^ + \frac{v_{22}}{v_{11}}(k(0) - k^*)$.*

Uniqueness of Equilibrium Path

Therefore, from (24), we can analytically obtain the optimal growth path as follows:

$$\begin{aligned} k(t) - k^* &= (k(0) - k^*) \exp(\omega_2 t) \text{ and} \\ c(t) - c^* &= \frac{v_{22}}{v_{12}} (k(0) - k^*) \exp(\omega_2 t) \\ &\left(= \frac{v_{22}}{v_{12}} (k(t) - k^*) \right) \end{aligned} \quad (26)$$

Proposition 1

There exists a unique competitive equilibrium path.

The social planner's problem

Social Planner's Problem

Consider an economy in which a social planner directly determines the pair of time paths $\{c(t), k(t)\}$ so as to maximize

$$\begin{aligned} \max \quad & \int_0^{\infty} \exp(-(\rho - n)t) u(c(t)) dt, \\ \text{s.t.} \quad & \dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t), \\ & k(0) > 0 \text{ given} \end{aligned}$$

(*) This is the “original” formulation by Cass (1965) and Koopmans (1965).

The time paths $\{c(t), k(t)\}$ that solves the above problem is called the **First-Best Allocation**.

Theorem

Theorem 3

The competitive equilibrium path achieves the first-best allocation.