Mathematical Preliminary (1) Infinite-Horizon Dynamic Optimization in Discrete Time

IEE.B402. Advanced Macroeconomics

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Plan

- Preliminary: two-period utility maximization problem
  $\rightarrow$ A key condition: Euler equation
- Infinite-horizon utility maximization problem
  - Two key conditions:
    Euler equation and the transversality condition (TVC)
  - Sufficiency of Euler eq. and TVC for utility maximization
- How to derive the utility-maximizing conditions
  - Lagrangian approach
  - Recursive approach using the Dynamic Programming technique
- Euler equation in the general problem
- A firm's present value maximization problem

Notation

Notation:

- Time is discrete, indexed by t
- c_t: consumption
- a_t: assets evaluated at the beginning of a period
- r_t: real interest rate
- w_t: real wage rate

Budget constraint in period t:

$$r_t a_t + w_t = c_t + \underbrace{a_{t+1} - a_t}_{\text{savings/borrowing}} .$$

$1 + r_t$: gross interest rate (i.e., interest rate incl. principal)

Preliminary: Two-Period Problem

Objective function $= u(c_0) + \beta u(c_1)$

- $\beta \in (0, 1)$: discount factor
- $u(\cdot)$: one-period utility function

Assumption 1

The function u is

1. *differentiable (of necessary times)*
2. *increasing: $u'(c) > 0$*
3. *strictly concave: $u''(c) < 0$*

Assumption 2

The function u satisfies the Inada condition: $\lim_{c \to 0^+} u'(c) = \infty$.

Preliminary: Two-Period Problem

A household's two-period utility maximization problem:

$$
\begin{aligned}
\max_{c_0, c_1, a_1, a_2} \quad & u(c_0) + \beta u(c_1) \\
\text{s.t.} \quad & (1 + r_0)a_0 + w_0 = c_0 + a_1, && (1) \\
& (1 + r_1)a_1 + w_1 = c_1 + a_2, && (2) \\
& a_2 \geq 0. && (3)
\end{aligned}
$$

($*$) A decision maker takes his/her initial assets a_0 and (r_t, w_t) as given.
($*$) The constraint $c_t \geq 0$ is omitted, because this never binds from Assumption 2.

Euler Equation and A Terminal Condition

Utility-maximizing conditions:

$$u'(c_0) = \beta u'(c_1)(1 + r_1), \tag{4}$$

$$a_2 = 0. \tag{5}$$

and (1)–(2)

- (4) is called the Euler equation.
- (5) means that the constraint (3) binds.

Intuition of (5) is quite simple: Under the situation that he/she is not allowed to leave a debt, it is optimal to consume out of his/her wealth.

From finite- to infinite-horizon problems

Objective Function and Feasible Plan

Let $\{x_t\}$ denote the infinite sequence of a variable x_t.

Objective function:

$$U(\{c_t\}) = \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t u(c_t). \tag{6}$$

Feasible set in period t:

$$\Omega_t(a_t) = \{(c, a') \mid (1 + r_t)a_t + w_t = c + a'\}. \tag{7}$$

Additional constraint:

$$\lim_{T\to\infty} \frac{a_{T+1}}{\prod_{j=1}^{T}(1 + r_j)} \geq 0. \tag{8}$$

- $\prod_{j=1}^{T}(1 + r_j) = (1 + r_1) \times (1 + r_2) \times \cdots \times (1 + r_T)$.
- (8) is called the No-Ponzi-Game condition (NPG)

Objective Function and Feasible Plan

The pair of sequences $\{c_t, a_{t+1}\}$ satisfying

- $(c_t, a_{t+1}) \in \Omega_t(a_t)$ for all t, and
- NPG (8),

is called the feasible plan or admissible plan.

It is assumed $U(\{c_t\})$ in (6) is well-defined: i.e., $\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t u(c_t)$ is finite for all feasible plans.

Infinite-Horizon Problem

Infinite-horizon utility maximization problem (UMP):

$$\begin{aligned}
\max_{\{c_t, a_{t+1}\}} \quad & U(\{c_t\}) = \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t.} \quad & a_{t+1} = (1 + r_t) a_t + w_t - c_t \quad t = 0, 1, 2, \ldots, \\
& \lim_{T \to \infty} \frac{a_{T+1}}{\prod_{j=1}^{T} (1 + r_j)} \geq 0.
\end{aligned} \tag{9}$$

Sufficient Conditions of Infinite-Horizon UMP

Theorem 1 (Sufficiency)

Suppose that $u(c)$ *satisfies Assumption 1. Then, given* a_0 *and* $\{r_t, w_t\}$*, the pair of sequences* $\{c_t, a_{t+1}\}$ *is the utility-maximizing plan if it satisfies (9) and*

$$u'(c_t) = \beta u'(c_{t+1})(1 + r_{t+1}) \quad t = 0, 1, 2, \ldots, \tag{10}$$

$$\lim_{T\to\infty} \beta^T u'(c_T) a_{T+1} = 0. \tag{11}$$

(10) is the Euler equation, and (11) is called the transversality condition (TVC).

Proof of Theorem 1

- Let $\{c_t^*, a_{t+1}^*\}$ denote the pair of sequences satisfying (9)–(11).
- Since u is strictly concave, we have the following inequality:

$$u(c_t^*) > u(c_t) + u'(c_t^*)(c_t^* - c_t),$$

  where c_t can be arbitrarily chosen from $\Omega_t(a_t)$ except c_t^*.
- Then, we have

$$\sum_{t=0}^{\infty} \beta^t u(c_t^*) - \sum_{t=0}^{\infty} \beta^t u(c_t) > D^*,$$

  where

$$D^* \equiv \sum_{t=0}^{\infty} \beta^t u'(c_t^*)(c_t^* - c_t).$$

  Therefore, we can show this theorem by showing $D^* \geq 0$.

Proof of Theorem 1

- Since both of $\{c_t^*\}$ and $\{c_t\}$ are feasible, D^* is rewritten as

$$D^* = \sum_{t=0}^{\infty} \beta^t u'(c_t^*) \left[(1+r_t)(a_t^* - a_t) - (a_{t+1}^* - a_{t+1})\right].$$

- Using the Euler equation (10), this is further reduced to

$$D^* = -\lim_{T\to\infty} \beta^T u'(c_T^*)(a_{T+1}^* - a_{T+1}).$$

- By its definition, $\{c_t^*, a_{t+1}^*\}$ satisfies the TVC, which implies that

$$D^* = \lim_{T\to\infty} \beta^T u'(c_T^*) a_{T+1}.$$

Proof of Theorem 1

- Using the Euler equation, we obtain

$$\beta^T u'(c_T^*) = u'(c_0^*) \frac{1}{\prod_{j=1}^{T}(1+r_j)}.$$

- Using the above equation, D^* is finally given by

$$D^* = u'(c_0^*) \times \lim_{T\to\infty} \frac{a_{T+1}}{\prod_{j=1}^{T}(1+r_j)}.$$

- Since the sequence $\{a_{t+1}\}$ is feasible, it satisfies the NPG. This shows that $D^* \geq 0$.

$\blacksquare$

Relationship btw NPG and TVC

Using the Euler equation, we can rewrite the TVC as follows:

$$\lim_{T\to\infty} \frac{a^*_{T+1}}{\prod_{j=1}^{T}(1+r_j)} = 0.$$

$\Downarrow$

- NPG is the constraint, which prohibits the household to leave a debt in a present value sense.
- TVC is the condition for maximization, which ensures the binding of the NPG.

How to derive the conditions

Lagrangian Approach

Cookbook Procedure

1. Construct the Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \lambda_t \left((1 + r_t) a_t + w_t - c_t - a_{t+1} \right) \right],$$

   where λ_t is the Lagrangian multiplier in period t.

2. Derive the first-order (necessary) conditions wrt c_t and a_{t+1}:

$$c_t : \quad u'(c_t) - \lambda_t = 0 \quad t = 0, 1, 2, \ldots \tag{12}$$

$$a_{t+1} : \quad -\lambda_t + \beta \lambda_{t+1} (1 + r_{t+1}) = 0 \quad t = 0, 1, 2, \ldots \tag{13}$$

   $\rightarrow$ From (12) and (13) we can obtain the Euler equation. We have already shown that this is also sufficient.

3. Derive the TVC as the equality holding condition of the NPG.

Recursive Approach using Dynamic Programming

We can derive the UMP conditions by using the "recursive feature" of the problem.

Suppose that the household reconsiders the following problem in period t_0:

$$\max_{\{c_t, a_{t+1}\}_{t=t_0}^{\infty}} \quad \sum_{t=t_0}^{\infty} \beta^{t-t_0} u(c_t)$$

$$\text{s.t.} \quad \{c_t, a_{t+1}\}_{t=t_0}^{\infty} \text{is feasible}$$

The initial value of assets is now given by $a_{t_0}^*$.

Theorem 2

Let $\{c_t^{**}, a_{t+1}^{**}\}_{t=t_0}^{\infty}$ denote the utility-maximizing pair of sequences which solves the above problem. Then, $c_t^{**} = c_t^*$ and $a_{t+1}^{**} = a_{t+1}^*$ for all $t \geq t_0$.

Proof.

Omitted

Value Function and Bellman Equation

Define the following function:

$$V_t(a_t) = \max_{\{c_\tau, a_{\tau+1}\}_{\tau=t}^{\infty}} \left\{ \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau) \;\middle|\; \{c_\tau, a_{\tau+1}\}_{\tau=t}^{\infty} \text{ is feasible} \right\}.$$

$\rightarrow V_t$ is called the value function.

We can convert the original problem to the two-period problem:

$$V_t(a_t) = \max_{c,a'} \{u(c) + \beta V_{t+1}(a') \mid (c, a') \in \Omega_t(a_t)\}. \tag{14}$$

(14) is called the Bellman equation.

Roughly speaking,
Recursive approach = Obtain $\{c_t^*, a_{t+1}^*\}$ by solving the Bellman equation (14), instead of solving the original problem.

Recursive Approach

Assume that $V_t(a_t)$ is differentiable.

Cookbook Procedure

1. Derive the F.O.C: $a_{t+1} = (1 + r_t)a_t + w_t - c_t$ and

$$u'(c_t) = \beta V'_{t+1}(a_{t+1}). \tag{15}$$

$\Rightarrow c_t$ is implicitly obtained. We let this denote $\phi_t(a_t)$.

$\Rightarrow a_{t+1} = \psi_t(a_t) \equiv (1 + r_t)a_t + w_t - \phi_t(a_t)$

2. Bellman equation must satisfy

$$V_t(a_t) = u(\phi_t(a_t)) + \beta V_{t+1}[\psi_t(a_t)].$$

3. Using the envelope theorem,

$$V'_t(a_t) = \beta V'_{t+1}(a_{t+1})(1 + r_t). \tag{16}$$

From (15) and (16) we obtain the Euler equation.

General problem

Terminology

- State variables: already given at the beginning of each period (e.g., assets)
- Control variables: immediately under the control (e.g., consumption)
- Let x_t and u_t respectively denote state and control variables.
- Transition equation:
$$x_{t+1} = g_t(x_t, u_t).$$
Assume that g is differentiable wrt x_t and u_t.

Terminlogy

- One-period return function: $F_t(x_t, u_t)$.
  - Assume that F is differentiable wrt x_t and u_t.
  - In Economics, F is often specified as

$$F_t(x_t, u_t) = \left(\prod_{j=1}^{t} \beta_j\right) f_t(x_t, u_t),$$

    with $\beta_j \in (0,1)$ and $\prod_{j=1}^{0} \beta_j$ is assigned to 1.
- Terminal constraint:

$$\lim_{T\to\infty} b_T x_T \geq \overline{x}.$$

  where b_T is a variable such that $\lim_{T\to\infty} b_T < \infty$ and $\overline{x}$ is a constant.

General Problem

Infinite-horizon maximization problem:

$$\max_{\{u_t, x_{t+1}\}} \quad \sum_{t=0}^{\infty} \left(\prod_{j=1}^{t} \beta_j \right) f_t(x_t, u_t)$$

$$\text{s.t} \quad x_{t+1} = g_t(x_t, u_t) \quad t = 0, 1, 2, \ldots,$$

$$\lim_{T \to \infty} b_T x_T \geq \overline{x}.$$

Euler equation:

$$\frac{f_{u,t}}{g_{u,t}} + \beta_{t+1} \left(f_{x,t+1} + \frac{f_{u,t+1}}{g_{u,t+1}} g_{x,t+1} \right) = 0,$$

where $f_{u,t} = \partial f(x_t, u_t)/\partial u_t$, and the same rule applies to the other expressions ($g_{u,t}$, $f_{x,t}$ and $g_{x,t}$).

Other application: A firm's dynamic decision making for investment

A firm's present value maximization

Setup:

- Consider a firm, of which production function is

$$Y_t = F(K_t, L_t).$$

  - Y_t : output; K_t: capital; and L_t: labor.
  - F satisfies the same properties as the slides in Jun. 15.
- Now, suppose that the firm does not rent capital, but invest by itself.
  - I_t: investment.
  - $K_{t+1} = I_t + (1-\delta)K_t, \ \delta \in (0,1)$.
- Net cash flow (NCF) in period t:

$$\pi_t = \underbrace{\underbrace{Y_t - w_t L_t}_{\text{gross profit}} - I_t}_{\text{NCF}}.$$

A firm's present value maximization

A firm's problem:

$$\max_{\{L_t, I_t, K_{t+1}\}} \quad \sum_{t=0}^{\infty} \left(\frac{1}{\prod_{j=1}^{t}(1+r_j)} \right) (F(K_t, L_t) - w_t L_t - I_t)$$

$$\text{s.t.} \quad K_{t+1} = I_t + (1-\delta)K_t \quad t = 0, 1, 2, \ldots,$$

$$\lim_{T\to\infty} \frac{K_{T+1}}{\prod_{j=1}^{T}(1+r_j)} \geq 0.$$

where $\prod_{j=1}^{0}(1+r_j)$ is assigned to 1.
$(*)$ Objective function $=$ Infinite sum of present value of NCF

Quiz

Derive the conditions of this firm's present value maximization problem.