# QIP Course 4: Quantum Teleportation

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# Answers of prev. exercises

$$|{-}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |{|}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, |\Psi\rangle = \frac{|{-}\rangle \otimes |{-}\rangle + |{|}\rangle \otimes |{|}\rangle}{\sqrt{2}}$$

$I$ is the $2 \times 2$ identity matrix. When you answer to the following, avoid expanding vectors into their components.

1. Show that the length (norm) of $|\Psi\rangle$ is 1.

$$
\begin{aligned}
& \langle\Psi|\Psi\rangle \\
= \quad & \frac{\langle -| \otimes \langle -| + \langle | | \otimes \langle | |}{\sqrt{2}} \cdot \frac{|-\rangle \otimes |-\rangle + | | \rangle \otimes | | \rangle}{\sqrt{2}} \\
= \quad & \frac{1}{2}(\langle -| \otimes \langle -||-\rangle \otimes |-\rangle + \langle | | \otimes \langle | || | \rangle \otimes | | \rangle \\
& + \langle -| \otimes \langle -|| | \rangle \otimes | | \rangle + \langle | | \otimes \langle | ||-\rangle \otimes |-\rangle) \\
= \quad & \frac{1}{2}(\langle -|-\rangle \times \underbrace{\langle -|-\rangle}_{=1} + \langle | | | \rangle \times \underbrace{\langle | | | \rangle}_{=1} + \langle -| | \rangle \times \underbrace{\langle -| | \rangle}_{=0} + \langle | |-\rangle \times \underbrace{\langle | |-\rangle}_{=0}) \\
= \quad & \frac{1}{2}(1 + 1 + 0 + 0) = 1
\end{aligned}
$$

2. Show that $X$ and $Z$ are unitary matrices.
Straightforward computation.

3. Express $(X\otimes I)|\Psi\rangle$ in terms of $|-\rangle$ and $|\,|\,\rangle$.

$$
\begin{aligned}
& (X\otimes I)|\Psi\rangle \\
= \quad & (X\otimes I)\frac{|-\rangle\otimes|-\rangle+|\,|\,\rangle\otimes|\,|\,\rangle}{\sqrt{2}} \\
= \quad & \frac{(X\otimes I)|-\rangle\otimes|-\rangle+(X\otimes I)|\,|\,\rangle\otimes|\,|\,\rangle}{\sqrt{2}} \\
= \quad & \frac{\overbrace{X|-\rangle}^{=|\,|\,\rangle}\otimes|-\rangle+\overbrace{X|\,|\,\rangle}^{=|-\rangle}\otimes|\,|\,\rangle}{\sqrt{2}} \\
= \quad & \frac{|\,|\,\rangle\otimes|-\rangle+|-\rangle\otimes|\,|\,\rangle}{\sqrt{2}}
\end{aligned}
$$

4. Express $(Z\otimes I)|\Psi\rangle$ in terms of $|-\rangle$ and $|\,|\,\rangle$.
Answer: Similar to the last question. $\frac{|-\rangle\otimes|-\rangle-|\,|\,\rangle\otimes|\,|\,\rangle}{\sqrt{2}}$

5. Suppose that one measures the observable $Z \otimes I$ of the system in the state $|\Psi\rangle$. For each measurement outcome, calculate the probability of getting the outcome and the state after measurement.

| outcome | probability | state |
|---|---|---|
| $+1$ | 0.5 | $|-\rangle \otimes |-\rangle$ |
| $-1$ | 0.5 | $||\rangle \otimes ||\rangle$ |

Projection can be easily computed, because …

Spectral decomposition of $Z = |-\rangle\langle -| - ||\rangle\langle ||$.
Spectral decomposition of $Z \otimes I = |-\rangle\langle -| \otimes I - ||\rangle\langle || \otimes I$.

$$(|-\rangle\langle -| \otimes I)\frac{|-\rangle \otimes |-\rangle + ||\rangle \otimes ||\rangle}{\sqrt{2}} = \frac{|-\rangle \overbrace{\langle -|-\rangle}^{=1} \otimes I|-\rangle + |-\rangle \overbrace{\langle -||\rangle}^{=0} \otimes I||\rangle}{\sqrt{2}}$$
$$= \frac{|-\rangle \otimes |-\rangle}{\sqrt{2}}$$

We can easily see that

| outcome | probability | state |
|---|---|---|
| $+1$ | 0.5 | $|-\rangle \otimes |-\rangle$ |

6. Is $Z \otimes Z$ a Hermitian matrix?
Yes.
7. Is $Z \otimes Z$ a unitary matrix?
Yes.
8. Write all the eigenvalues of $Z \otimes Z$ and an orthonormal basis of each eigenspace. After that, compute the spectral decomposition of $Z \otimes Z$.
Since the eigenvalues of $Z$ are $+1$ and $-1$, the eigenvalues of $Z \otimes Z$ are $+1$ and $-1$. The spectral decomposition of $Z \otimes Z$ is

$$
\begin{aligned}
&(+1)(|-\rangle\langle -| \otimes |-\rangle\langle -| + |\,|\,\rangle\langle\,|\,| \otimes |\,|\,\rangle\langle\,|\,|) \\
&+(-1)(|-\rangle\langle -| \otimes |\,|\,\rangle\langle\,|\,| + |\,|\,\rangle\langle\,|\,| \otimes |-\rangle\langle -|)
\end{aligned}
$$

An orthonormal basis (ONB) of eigenspace belonging to eigenvalue $+1$ is $\{|-\rangle \otimes |-\rangle, |\,|\,\rangle \otimes |\,|\,\rangle\}$, An ONB of eigenspace belonging to eigenvalue $-1$ is $\{|-\rangle \otimes |\,|\,\rangle, |\,|\,\rangle \otimes |-\rangle\}$.

9. Answer Question 5 with $Z \otimes I$ replaced with $Z \otimes Z$.
Since

$$
\begin{aligned}
(|-\rangle\langle-| \otimes |-\rangle\langle-| + |\,|\,\rangle\langle\,|\,| \otimes |\,|\,\rangle\langle\,|\,|)|\Psi\rangle &= |\Psi\rangle, \\
(|-\rangle\langle-| \otimes |\,|\,\rangle\langle\,|\,| + |\,|\,\rangle\langle\,|\,| \otimes |-\rangle\langle-|)|\Psi\rangle &= 0,
\end{aligned}
$$

| outcome | probability | state |
|---|---|---|
| $+1$ | 1 | $\lvert\Psi\rangle$ |
| $-1$ | 0 | |

Because there are only two eigenvalues of $Z \otimes Z$, the number of measurement outcomes is TWO.

10 (Optional). Prove that Eq. (10) is the spectral decomposition of $A \otimes B$. You must calculate the set of eigenvalues of $A \otimes B$ and the projectors onto its eigenspaces. It is not enough to simply prove the equality in Eq. (10).
Answer: See the next several pages.

# Eigenvalues and spectral decomposition of $A \otimes B$

$A, B$: $\ell \times \ell$ Hermitian or unitary matrices.

$$\begin{aligned} A &= \lambda_1 P_1 + \cdots + \lambda_m P_m, \\ B &= \eta_1 Q_1 + \cdots + \eta_n Q_n. \end{aligned}$$

Firstly we compute the set of eigenvalues of $A \otimes B$.

Let $A|\varphi\rangle = \lambda|\varphi\rangle$, $B|\psi\rangle = \eta|\psi\rangle$. Then we have

$$(A \otimes B)(|\varphi\rangle \otimes |\psi\rangle) = (A|\varphi\rangle) \otimes (B|\psi\rangle) = \lambda\eta|\varphi\rangle \otimes |\psi\rangle,$$

and $|\varphi\rangle \otimes |\psi\rangle$ belongs to the eigenvalue $\lambda\eta$ of $A \otimes B$. Therefore $\lambda_i \eta_j$ is an eigenvalue of $A \otimes B$ for all $i, j$.

$|\varphi_1\rangle, \ldots, |\varphi_\ell\rangle$: orthogonal eigenvectors of $A$
$|\psi_1\rangle, \ldots, |\psi_\ell\rangle$: orthogonal eigenvectors of $B$.
The above eigenvectors must belong to some eigenvalues of $A$ or $B$, and the dimension of the space spanned by $|\varphi_i\rangle \otimes |\psi_j\rangle$ for $i, j = 1, \ldots, \ell$ is $\ell^2$.

On the other hand, I want to prove that there is no other eigenvalue other than $\{\lambda_i \eta_j \mid i = 1, \ldots, m, j = 1, \ldots, n\}$ by a contradiction. If there is another eigenvalue, then there exists an eigenvector linearly independent of $|\varphi_i\rangle \otimes |\psi_j\rangle$ and the number of linearly independent eigenvectors is larger than $\ell^2$, which is a contradiction. We have prove that the set of eigenvalues is $\{\lambda_i \eta_j \mid i = 1, \ldots, m, j = 1, \ldots, n\}$.

# Computation of projector

Let $\alpha$ be an eigenvalue of $A \otimes B$. We will compute the projector onto the eigenspace belonging to $\alpha$.

By change of index, we may assume that $\alpha = \lambda_1\eta_1 = \lambda_2\eta_2 = \cdots = \lambda_u\eta_u$ and $\alpha \neq \lambda_i\eta_j$ unless $i = j \leq u$.

We have to proved that $P_1 \otimes Q_1 + \cdots + P_u Q_u$

is the projection onto the eigenspace belonging to $\alpha$.

$\{|\varphi_{i1}\rangle, |\varphi_{i2}\rangle, \ldots\}$: An ONB of the eigenspace of $A$ belonging to $\lambda_i$

$\{|\psi_{j1}\rangle, |\varphi_{j2}\rangle, \ldots\}$: An ONB of the eigenspace of $B$ belonging to $\eta_j$

An ONB of the eigenspace belonging to $\alpha$ is

$$\{|\varphi_{ij}\rangle \otimes |\psi_{ik}\rangle \mid i = 1, \ldots, u, j, k \geq 1\}.$$

Therefore its projector is

$$\begin{aligned}
\sum_{i,j,k} |\varphi_{ij}\rangle\langle\varphi_{ij}| \otimes |\psi_{ik}\rangle\langle\psi_{ik}| &= \sum_i \sum_{j,k} |\varphi_{ij}\rangle\langle\varphi_{ij}| \otimes |\psi_{ik}\rangle\langle\psi_{ik}| \\
&= \sum_i \left( \sum_j |\varphi_{ij}\rangle\langle\varphi_{ij}| \right) \otimes \left( \sum_k |\psi_{ik}\rangle\langle\psi_{ik}| \right) \\
&= \sum_i P_i \otimes Q_i.
\end{aligned}$$

Therefore, the spectral decomposition of $A \times B$ is given by

$$\sum_{\alpha \in \{\lambda_i \eta_j | 1 \le i \le m, 1 \le j \le n\}} \alpha \sum_{\lambda_i \eta_j = \alpha} P_i \otimes Q_j,$$

# What is quantum information processing?

The research of quantum information explores what can be done under the assumption that all the unitary matrices, all the states, and all the measurements are physically realizable.

We (at least I) do not care how one can implement (realize) a given unitary matrix by a physical device.

**However,** the research of quantum computation imposes some restrictions on the set of available unitary matrices because otherwise any computation can be done by a single unitary matrix and we become unable to consider the computational complexity of quantum computation.

# Some notations

$$
\begin{aligned}
& |\varphi\rangle \otimes |\psi\rangle \\
= \quad & |\varphi\rangle|\psi\rangle \\
= \quad & |\varphi\psi\rangle
\end{aligned}
$$

A two-dimensional quantum system is said to be a *qubit*.
Qubit represents QUantum BIT.

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

# Quantum teleportation

Draw a figure on a blackboard.

- Send a quantum state to a recipient who is spacially apart from the sender.
- The sender DOES NOT send the physical system.
- The sender sends 2 bits information for transmission of 1 qubit.
- The sender and the receiver share the entangled state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Example: Suppose that the sender is on the earth, and the receiver is in a spaceship far apart from the earth.
A physical object is reproduced at a distant place without sending a physical object. Doesn't it seem like a teleportation??

# Controlled NOT

Manipulation of a quantum system is expressed by a unitary matrix.
A unitary matrix $U$ can be specified by $U|\varphi\rangle$ for every basis vector $|\varphi\rangle$

$U$: $4 \times 4$ unitary matrix

$$U|00\rangle = |00\rangle, \quad U|01\rangle = |01\rangle,$$

$$U|10\rangle = |11\rangle, \quad U|11\rangle = |10\rangle.$$

The right qubit is negated iff the left qubit is one.
$U$ is similar to the NOT gate on the second qubit controlled by the first qubit.

left qubit: control qubit of CNOT
right qubit: target qubit of CNOT

# Teleportation (1)

$|\Psi\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$ is shared.
$\alpha|0\rangle + \beta|1\rangle$ is to be sent.

The state of the total system is

$$
\begin{aligned}
& (\alpha|0\rangle + \beta|1\rangle)|\Psi\rangle \\
= \quad & \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\right]
\end{aligned}
$$

The sender has the leftmost and the middle qubits, and the receiver has the rightmost qubit.

# Teleportation (2)

$$\frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\right]$$

Applying CNOT with
control quibit: leftmost qubit
target qubit: middle qubit

$$\frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\right]$$

# Teleportation (3)

$$\frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\right]$$

The matrix $H$:

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \tag{1}$$

Applying $H$ to the leftmost qubit:

$$\begin{aligned} & \frac{1}{2}[\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \\ & \quad \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)] \\ = \; & \frac{1}{2}[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + \\ & \quad |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)] \end{aligned}$$

# Teleportation (4)

$$\frac{1}{2}[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)] \tag{2}$$

The sender measures the observable $Z_1$ of the leftmost qubit, and the $Z_2$ of the middle qubit, where

$$Z_1 = Z_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

# Teleportation (5)

The sender sends the measurement outcomes, and the receiver applies the following unitary matrix to the rightmost qubit according to outcomes.

| $Z_1$ | $Z_2$ | Receiver's matrix |
|---|---|---|
| $+1$ | $+1$ | $2 \times 2$ identity matrix |
| $+1$ | $-1$ | $X$ |
| $-1$ | $+1$ | $Z$ |
| $-1$ | $-1$ | $ZX$ |

Then $\alpha|0\rangle + \beta|1\rangle$ is teleported to the receiver (Exercise).

# Explanation of teleportation

The sender has this qubit and this qubit. The receiver has this qubit.

Before teleportation:

$$(\alpha|0\rangle + \beta|1\rangle)\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

After teleportation:

$$|??\rangle(\alpha|0\rangle + \beta|1\rangle),$$

where ? is 0 or 1, depending on the measurement outcomes.

# Exercise (60 min.??)

Please discuss them with other students. You are also welcomed to talk with the lecturer.

1. For each measurement outcome $(\pm 1, \pm 1)$ of $(Z_1, Z_2)$, compute the probability of getting the outcome and the state of three qubit after measurement of the state (2). Write your derivation of answer in detail for at least one measurement outcome. Answer in the following format:

| $(Z_1, Z_2)$ | probability | state |
|---|---|---|
| $(+1, +1)$ | ? | ? |
| $(+1, -1)$ | ? | ? |
| $(-1, +1)$ | ? | ? |
| $(-1, -1)$ | ? | ? |

# Hint for Q1

Q1 can be solved in the follwing steps for $(+1, -1)$.

1. Compute the spectral decomposition of the observable $Z_1 \otimes I_{2\times 2} \otimes I_{2\times 2}$. Let $P^{(1)}_{+1}$ be the projection for the eigenvalue $+1$.
2. Compute $P^{(1)}_{+1}$·state of (2) / $\|P^{(1)}_{+1}$·state of (2) $\|$.
3. Compute the spectral decomposition of the observable $I_{2\times 2} \otimes Z_2 \otimes I_{2\times 2}$. Let $P^{(2)}_{-1}$ be the projection for the eigenvalue $-1$.
4. Compute $P^{(2)}_{-1}$ times the quantum state obtained in Step 2.

**Please make sure that the sum of probabilities is 1 in your answer.**

2. For each measurement outcome, compute the state of three qubits after the receiver applies the matrix to the rightmost qubit. Write your derivation of answer in detail for at least one measurement outcome. Answer in the following format:

| $(Z_1, Z_2)$ | state |
|---|---|
| $(+1, +1)$ | ? |
| $(+1, -1)$ | ? |
| $(-1, +1)$ | ? |
| $(-1, -1)$ | ? |

**Your understanding only comes through mathematics and your hand computation, because the intuition of human being is useless and misleading in the quantum physics!**