QIP Course 3: Basics of QIP (Part 2)

Ryutaroh Matsumoto

Nagoya University, Japan
Send your comments to ryutaroh.matsumoto@nagoya-u.jp

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@ Tokyo Tech.

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The slides include figures from the http://openstaxcollege.org/ and the Wikipedia.

Contents of the slides

- Manipulation of quantum states
- Composite system
- Entangled state

Answers of prev. exercises

1. Yes.
2. $$X = +1 \cdot \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + (-1) \cdot \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$
3.

outcome	probability	state
$+1$	0.5	$\frac{1}{2}\begin{pmatrix} 1+i \\ 1+i \end{pmatrix}$
-1	0.5	$\frac{1}{2}\begin{pmatrix} 1-i \\ i-1 \end{pmatrix}$

4. The observable X distinguishes the / and \ polarizations.

Orthonormal basis (Preparation for Q5)

$\{|\varphi_1\rangle, \ldots, |\varphi_n\rangle\}$: an orthonormal basis of V.
$|\psi\rangle \in V$ can be written as

$$a_1|\varphi_1\rangle + \cdots + a_n|\varphi_n\rangle.$$

We have

$$\left(\sum_{i=1}^{n} |\varphi_i\rangle\langle\varphi_i|\right)|\psi\rangle = \sum_{i=1}^{n} |\varphi_i\rangle\langle\varphi_i|\psi\rangle = \sum_{i=1}^{n} |\varphi_i\rangle a_i = |\psi\rangle.$$

Thus

$$\sum_{i=1}^{n} |\varphi_i\rangle\langle\varphi_i| = I. \tag{1}$$

Assume $i \neq j$.

$$\begin{aligned} |\varphi_i\rangle\langle\varphi_i||\varphi_j\rangle\langle\varphi_j| &= 0, & (2)\\ |\varphi_i\rangle\langle\varphi_i||\varphi_i\rangle\langle\varphi_i| &= |\varphi_i\rangle\langle\varphi_i|, & (3)\\ (|\varphi_i\rangle\langle\varphi_i|)^* &= (\langle\varphi_i|)^*(|\varphi_i\rangle)^* = |\varphi_i\rangle\langle\varphi_i|. & (4) \end{aligned}$$

Properties of a projector

$$P_1 = |\varphi_1\rangle\langle\varphi_1| + \cdots + |\varphi_m\rangle\langle\varphi_m|$$
$$P_2 = |\varphi_{m+1}\rangle\langle\varphi_{m+1}| + \cdots + |\varphi_n\rangle\langle\varphi_n|$$

$$P_1^* = P_1 \text{ (by Eq. (4))} \tag{5}$$
$$P_1 P_2 = 0 \text{ (by Eq. (2))} \tag{6}$$
$$P_1 P_1 = P_1 \text{ (by Eqs. (2) and (3))} \tag{7}$$

Answer to Q5

5. Prove

$$\sum_{j=1}^{n} \left\| P_j |\varphi\rangle \right\|^2 = 1.$$

A: Hermitian matrix

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n$$

Two eigenvectors belonging to different eigenvalues are orthogonal (see your linear algebra textbook).

$$\Downarrow$$

There exists an orthonormal basis $\{|\psi_{ij}\rangle\}$ such that $\{|\psi_{i1}\rangle, \ldots, |\psi_{im_i}\rangle\}$ is an orthonormal basis of the eigenspace belonging to λ_i.

$$\Downarrow$$

$$P_i = |\psi_{i1}\rangle\langle\psi_{i1}| + \cdots + |\psi_{im_i}\rangle\langle\psi_{im_i}|.$$

$$\begin{aligned}
P_i^* &= P_i \text{ (by Eq. (5))}, \\
P_i P_j &= \begin{cases} P_i (i = j), \\ 0 (i \neq j) \end{cases} \text{ (by Eqs. (6) and (7))}, \\
P_1 + \cdots + P_n &= I \text{ (by Eq. (1))}.
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{n} \| P_i |\varphi\rangle \|^2 &= \sum_{i=1}^{n} \langle\varphi| P_i^* P_i |\varphi\rangle = \sum_{i=1}^{n} \langle\varphi| P_i P_i |\varphi\rangle \\
&= \sum_{i=1}^{n} \langle\varphi| P_i |\varphi\rangle = \langle\varphi| \left(\sum_{i=1}^{n} P_i \right) |\varphi\rangle \\
&= \langle\varphi|\varphi\rangle = 1.
\end{aligned}$$

Answer to Q6

Prove that Eq. (1) (defining P_W) is the projection onto W in the sense of the previous unit of this course.

V: linear space
W: linear subspace of V
$W^\perp$: orthogonal complement of W in V
One has to prove

1. $P_W|\varphi\rangle \in W$ for any $|\varphi\rangle$, and
2. $|\varphi\rangle - P_W|\varphi\rangle \in W^\perp$ for any $|\varphi\rangle$.

$\{|\psi_1\rangle, \ldots, |\psi_m\rangle\}$: orthonormal basis of W
$\{|\psi_{m+1}\rangle, \ldots, |\psi_n\rangle\}$: orthonormal basis of $W^\perp$

$$
\begin{aligned}
P_W &= \sum_{i=1}^{m} |\psi_i\rangle\langle\psi_i|, \\
P_{W^\perp} &= \sum_{i=m+1}^{n} |\psi_i\rangle\langle\psi_i|.
\end{aligned}
$$

$$
\begin{aligned}
P_W + P_{W^\perp} &= I, \text{ (by Eq. (1))} \\
|\varphi\rangle &= P_W|\varphi\rangle + P_{W^\perp}|\varphi\rangle
\end{aligned}
$$

$$
\begin{aligned}
P_W|\varphi\rangle &= \sum_{i=1}^{m} |\psi_i\rangle\langle\psi_i|\varphi\rangle \in W, \\
|\varphi\rangle - P_W|\varphi\rangle = P_{W^\perp}|\varphi\rangle &= \sum_{i=m+1}^{n} |\psi_i\rangle\langle\psi_i|\varphi\rangle \in W^\perp
\end{aligned}
$$

Observe that the above two equalities are what we had to verify.

Manipulation of a quantum system

Manipulation of a quantum system *without extracting information* is represented by a unitary matrix U.
A unitary matrix U is a matrix such that $UU^* = I$.

Example:

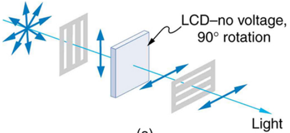
$$|{-}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |{|}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

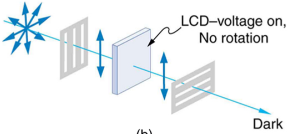
$$\begin{aligned} XX^* &= I, \\ ZZ^* &= I, \\ X|{-}\rangle &= |{|}\rangle, \\ X|{|}\rangle &= |{-}\rangle, \\ Z|{-}\rangle &= |{-}\rangle, \\ Z|{|}\rangle &= -|{|}\rangle, \end{aligned}$$

Example with polarization

The below is just rotating, neither X or Z. Source: OpenStax College.



(a)



(b)

(c)

Tenser product, or Kronecker product

A: $m \times n$ matrix, B: $p \times q$ matrix

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{pmatrix}$$

The tensor product of column vectors is defined by regarding column vectors as $m \times 1$ and $p \times 1$ matrices.
The tensor product of row vectors is similarly defined.

Properties of tensor products

α: a complex number

$$\begin{aligned} \alpha(|\varphi\rangle \otimes |\psi\rangle) &= (\alpha|\varphi\rangle) \otimes |\psi\rangle \\ &= |\varphi\rangle \otimes (\alpha|\psi\rangle) \\ (|\varphi_1\rangle + |\varphi_2\rangle) \otimes |\psi\rangle &= |\varphi_1\rangle \otimes |\psi\rangle + |\varphi_2\rangle \otimes |\psi\rangle \\ |\varphi\rangle \otimes (|\psi_1\rangle + |\psi_2\rangle) &= |\varphi\rangle \otimes |\psi_1\rangle + |\varphi\rangle \otimes |\psi_2\rangle \end{aligned}$$

(similar relations hold for matrices)

$$\begin{aligned} (A \otimes B)(|\varphi\rangle \otimes |\psi\rangle) &= (A|\varphi\rangle) \otimes (B|\psi\rangle) \\ &= A|\varphi\rangle \otimes B|\psi\rangle \\ (\langle\varphi_1| \otimes \langle\varphi_2|)(|\psi_1\rangle \otimes |\psi_2\rangle) &= \langle\varphi_1|\psi_1\rangle \cdot \langle\varphi_2|\psi_2\rangle \\ (A \otimes B)^* &= A^* \otimes B^* \\ (A \otimes B)^{-1} &= A^{-1} \otimes B^{-1} \end{aligned}$$

V, W: linear spaces

$V \otimes W$: linear space **spanned by** $\{|\varphi\rangle \otimes |\psi\rangle : |\varphi\rangle \in V, |\psi\rangle \in W\}$.

Tensor product is not the direct product

$\dim V \otimes W = \dim V \times \dim W$, while
$\dim V \times W = \dim V + \dim W$.

Suppose that $\vec{v} \in \mathbf{C}^2$ and $\vec{w} \in \mathbf{C}^3$.
The direct product $(\vec{v}, \vec{w})$ has 5 numbers as its components,
while the tensor product $\vec{v} \otimes \vec{w}$ has 6 numbers as its component.

Composite system

A quantum system 1 is represented by a linear space $\mathcal{H}_1$.
A quantum system 2 is represented by a linear space $\mathcal{H}_2$.

The quantum system consisting of system 1 and system 2 is represented by a vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Applying a unitary operator U_1 to system 1 is equivalent to applying $U_1 \otimes I$ to the composite system.

Measuring an observable A_1 of system 1 is equivalent to measuring the observable $A_1 \otimes I$ of the composite system.

Entangled state

V, W: linear space

Some vector in $V \otimes W$ cannot be written as $|\varphi\rangle \otimes |\psi\rangle$ for any $|\varphi\rangle \in V$ and $|\psi\rangle \in W$.

$$\frac{1}{\sqrt{2}}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} \tag{9}$$

If $ac \neq 0$ and $bd \neq 0$, then $a \neq 0$, $b \neq 0$, $c \neq 0$, and $d \neq 0$. Therefore, Eq. (8) cannot be expressed as Eq. (9).

A quantum state that cannot be expressed as $|\varphi\rangle \otimes |\psi\rangle$ is called an **entangled state**.

Subsystem of a composite system

A composite system consists of systems 1 and 2 is in an entangled state.

$$\Downarrow$$

The state of system 1 cannot be expressed by a state vector.

The state vector is an incomplete expression of quantum states.

$\Downarrow$ But

- Any quantum state can always be expressed as a state vector of some larger system ("purification" in unit 7).
- Matrix expression of the quantum state does not have such a drawback ("partial trace" in unit 6).

Spectral decomposition of a tensor product

A, B: Hermitian matrices
Spectral decompositions of A and B:

$$\begin{aligned} A &= \lambda_1 P_1 + \cdots + \lambda_m P_m, \\ B &= \eta_1 Q_1 + \cdots + \eta_n Q_n. \end{aligned}$$

The spectral decomposition of $A \otimes B$ is given by

$$A \otimes B = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \eta_j P_i \otimes Q_j. \tag{10}$$

Provide an example on the black board.
From the above equation, we can see that the set of eigenvalues of $A \otimes B$ is $\{\lambda_i \eta_j \mid i = 1, \ldots, m, j = 1, \ldots, n\}$.

Exercises (60 min.?)

Please discuss them with other students. You are also welcomed to talk with the lecturer.

You are not forced to solve the following problems, but you must able to quickly solve them, in order to follow the subsequent lectures.

$$|-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\,|\,\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, |\Psi\rangle = \frac{|-\rangle \otimes |-\rangle + |\,|\,\rangle \otimes |\,|\,\rangle}{\sqrt{2}}$$

I is the 2×2 identity matrix. When you answer to the following, avoid expanding vectors into their components, and insteadly use the equalities in p. 14 as much as possible.

1. Show that the length (norm) of $|\Psi\rangle$ is 1.
2. Show that X and Z are unitary matrices.
3. Express $(X \otimes I)|\Psi\rangle$ in terms of $|-\rangle$ and $|\,|\,\rangle$. Hint: Use relations in p. 14.
4. Express $(Z \otimes I)|\Psi\rangle$ in terms of $|-\rangle$ and $|\,|\,\rangle$.

5. Suppose that one measures the observable $Z \otimes I$ of the system in the state $|\Psi\rangle$. For each measurement outcome, calculate the probability of getting the outcome and the state after measurement.
6. Is $Z \otimes Z$ a Hermitian matrix?
7. Is $Z \otimes Z$ a unitary matrix?
8. Write all the eigenvalues of $Z \otimes Z$, an orthonormal basis of each eigenspace, and compute the spectral decomposition of $Z \otimes Z$.
9. Answer Question 5 with $Z \otimes I$ replaced with $Z \otimes Z$.
10 (Optional for non-math students). Prove that Eq. (10) is the spectral decomposition of $A \otimes B$. You must calculate the set of eigenvalues of $A \otimes B$ and the projectors onto its eigenspaces. It is not enough to simply prove the equality in Eq. (10).