Linear Ordinary Differential Equations

Preliminaries: Matrix Norms. Denote by $\mathrm{M}_n(\mathbb{R})$ the set of $n \times n$ matrix with real components, which can be identified the vector space $\mathbb{R}^{n^2}$. In particular, the Euclidean norm of $\mathbb{R}^{n^2}$ induces a norm

$$|X|_{\mathrm{E}} = \sqrt{\operatorname{tr}({}^t XX)} = \sqrt{\sum_{i,j=1}^{n} x_{ij}^2} \tag{1.1}$$

on $\mathrm{M}_n(\mathbb{R})$. On the other hand, we let

$$|X|_{\mathrm{M}} := \sup\left\{\frac{|X\boldsymbol{v}|}{|\boldsymbol{v}|}\,;\,\boldsymbol{v} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}\right\}, \tag{1.2}$$

where $|\cdot|$ on the right-hand side denotes the Euclidean norm of $\mathbb{R}^n$.

Lemma 1.1. (1) *The map $X \mapsto |X|_{\mathrm{M}}$ is a norm of $\mathrm{M}_n(\mathbb{R})$.*

(2) *For X, $Y \in \mathrm{M}_n(\mathbb{R})$, it holds that $|XY|_{\mathrm{M}} \leqq |X|_{\mathrm{M}}\,|Y|_{\mathrm{M}}$.*

(3) *Let $\lambda = \lambda(X)$ be the maximum eigenvalue of semi-positive definite symmetric matrix ${}^t XX$. Then $|X|_{\mathrm{M}} = \sqrt{\lambda}$ holds.*

(4) $(1/\sqrt{n})|X|_{\mathrm{E}} \leqq |X|_{\mathrm{M}} \leqq |X|_{\mathrm{E}}$.

(5) *The map $|\cdot|_{\mathrm{M}}\colon \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ is continuous with respect to the Euclidean norm.*

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Proof. Since $|X\boldsymbol{v}|/|\boldsymbol{v}|$ is invariant under scalar multiplications to $\boldsymbol{v}$, we have $|X|_{\mathrm{M}} = \sup\{|X\boldsymbol{v}|\,;\,\boldsymbol{v} \in S^{n-1}\}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Here, the $S^{n-1} \ni \boldsymbol{x} \mapsto |A\boldsymbol{x}| \in \mathbb{R}$ is a continuous function defined on a compact space, and so the map takes maximum. Thus, the right-hand side of (1.2) is well-defined. It is easy to verify that $|\cdot|_{\mathrm{M}}$ satisfies the axiom of the norm.

Since $A := {}^t XX$ is positive semi-definite the eigenvalues λ_j ($j = 1, \dots, n$) are non-negative real numbers. In particular, there exists an orthonormal basis $[\boldsymbol{a}_j]$ of $\mathbb{R}^n$ satisfying $A\boldsymbol{a}_j = \lambda_j \boldsymbol{a}_j$ ($j = 12, \dots, n$). Let λ be the maximum eigenvalues of A, and write $\boldsymbol{v} = v_1\boldsymbol{a}_1 + \dots + v_n\boldsymbol{a}_n$. Then it holds that

$$\langle X\boldsymbol{v}, X\boldsymbol{v}\rangle = \lambda_1 v_1^2 + \dots + \lambda_n v_n^2 \leqq \lambda\,\langle \boldsymbol{v}, \boldsymbol{v}\rangle\,,$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product of $\mathbb{R}^n$. The equality of this inequality holds if and only if $\boldsymbol{v}$ is the λ-eigenvector, proving (3). Noticing the norm (1.1) is invariant under conjugations $X \mapsto {}^t PXP$ ($P \in \mathrm{O}(n)$), we obtain $|X|_{\mathrm{E}} = \sqrt{\lambda_1 + \dots + \lambda_n}$ by diagonalizing ${}^t XX$ by an orthogonal matrix P. Then we obtain (4). Hence two norms $|\cdot|_{\mathrm{E}}$ and $|\cdot|_{\mathrm{M}}$ induce the same topology $\mathrm{M}_n(\mathbb{R})$. In particular, we have (5). $\square$

Preliminaries: Matrix-valued Functions.

Lemma 1.2. *Let X and Y be C^∞-maps defined on a domain $U \subset \mathbb{R}^m$ into $\mathrm{M}_n(\mathbb{R})$. Then*

(1) $\dfrac{\partial}{\partial u_j}(XY) = \dfrac{\partial X}{\partial u_j}Y + X\dfrac{\partial Y}{\partial u_j}$,

(2) $\dfrac{\partial}{\partial u_j}\det X = \operatorname{tr}\left(\widetilde{X}\dfrac{\partial X}{\partial u_j}\right)$, *and*

(3) $\dfrac{\partial}{\partial u_j}X^{-1} = -X^{-1}\dfrac{\partial X}{\partial u_j}X^{-1}$,

where $\widetilde{X}$ *is the cofactor matrix of* X*, and we assume in* (3).

Proposition 1.3. *Assume two* C^∞ *matrix-valued functions* $X(t)$ *and* $\Omega(t)$ *satisfy*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0. \tag{1.3}$$

Then

$$\det X(t) = (\det X_0)\exp\int_{t_0}^{t}\operatorname{tr}\Omega(\tau)\,d\tau \tag{1.4}$$

holds. In particular, if $X_0 \in \mathrm{GL}(n,\mathbb{R})$, [1] *then* $X(t) \in \mathrm{GL}(n,\mathbb{R})$ *for all* t.

Proof. By (2) of Lemma 1.2, we have

$$\begin{aligned}\frac{d}{dt}\det X(t) &= \operatorname{tr}\left(\widetilde{X}(t)\frac{dX(t)}{dt}\right) = \operatorname{tr}\left(\widetilde{X}(t)X(t)\Omega(t)\right)\\ &= \operatorname{tr}\bigl(\det X(t)\Omega(t)\bigr) = \det X(t)\operatorname{tr}\Omega(t).\end{aligned}$$

Here, we used the relation $\widetilde{X}X = X\widetilde{X} = (\det X)\,\mathrm{id}$[2]. Hence $\frac{d}{dt}\bigl(\rho(t)^{-1}\det X(t)\bigr) = 0$, where $\rho(t)$ is the right-hand side of (1.4). $\square$

[1] $\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\,\det A \neq 0\}$: the general linear group.

[2] In this lecture, id denotes the identity matrix.

Proposition 1.4. *Assume* $\Omega(t)$ *in* (1.3) *is skew-symmetric for all* t*, that is,* ${}^t\Omega + \Omega$ *is identically* O*. If* $X_0 \in \mathrm{O}(n)$ *(resp.* $X_0 \in \mathrm{SO}(n)$*)*[3]*,* $X(t) \in \mathrm{O}(n)$ *(resp.* $X(t) \in \mathrm{SO}(n)$*) for all* t.

Proof. By (1) in Lemma 1.2,

$$\begin{aligned}\frac{d}{dt}(X\,{}^tX) &= \frac{dX}{dt}{}^tX + X\,{}^t\!\left(\frac{dX}{dt}\right)\\ &= X\Omega\,{}^tX + X\,{}^t\Omega\,{}^tX = X(\Omega + {}^t\Omega)\,{}^tX = O.\end{aligned}$$

Hence $X\,{}^tX$ is constant, that is, if $X_0 \in \mathrm{O}(n)$,

$$X(t)\,{}^tX(t) = X(t_0)\,{}^tX(t_0) = X_0\,{}^tX_0 = \mathrm{id}\,.$$

if $X_0 \in \mathrm{O}(n)$, proves the first case of the proposition. Since $\det A = \pm 1$ when $A \in \mathrm{O}(n)$, the second case follows by continuity of $\det X(t)$. $\square$

Preliminaries: Norms of Matrix-Valued functions. Let $I = [a,b]$ be a closed interval, and denote by $C^0(I,\mathrm{M}_n(\mathbb{R}))$ the set of continuous functions $X\colon I \to \mathrm{M}_n(\mathbb{R})$. For any fixed number k, we define

$$\|X\|_{I,k} := \sup\left\{e^{-kt}|X(t)|_{\mathrm{M}}\,;\,t \in I\right\} \tag{1.5}$$

for $X \in C^0(I,\mathrm{M}_n(\mathbb{R}))$. When $k = 0$, $\|\cdot\|_{I,0}$ is the uniform norm for continuous functions, which is complete. Similarly, one can prove the following in the same way:

[3] $\mathrm{O}(n) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\,{}^tAA = A\,{}^tA = \mathrm{id}\}$: the orthogonal group; $\mathrm{SO}(n) = \{A \in \mathrm{O}(n)\,;\,\det A = 1\}$: the special orthogonal group.

Lemma 1.5. *The map* $\|\cdot\|_{I,k}\colon C^0(I, \mathrm{M}_n(\mathbb{R}))$ *is a complete norm.*

Linear Ordinary Differential Equations. We prove the fundamental theorem for *linear* ordinary differential equations.

Proposition 1.6. *Let* $\Omega(t)$ *be a* C^∞*-function valued in* $\mathrm{M}_n(\mathbb{R})$ *defined on an interval* I*. Then for each* $t_0 \in I$*, there exists the unique matrix-valued* C^∞*-function* $X(t) = X_{t_0,\mathrm{id}}(t)$ *such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = \mathrm{id}\,. \tag{1.6}$$

Proof. Uniqueness: Assume $X(t)$ and $Y(t)$ satisfy (1.6). Then

$$\begin{aligned} Y(t) - X(t) &= \int_{t_0}^t \bigl(Y'(\tau) - X'(\tau)\bigr)\,d\tau \\ &= \int_{t_0}^t \bigl(Y(\tau) - X(\tau)\bigr)\Omega(\tau)\,d\tau \end{aligned}$$

holds. Hence for an arbitrary closed interval $J \subset I$,

$$\begin{aligned} |Y(t) - X(t)|_{\mathrm{M}} &\leqq \left|\int_{t_0}^t \bigl|\bigl(Y(\tau) - X(\tau)\bigr)\Omega(\tau)\bigr|_{\mathrm{M}}\,d\tau\right| \\ &\leqq \left|\int_{t_0}^t |Y(\tau) - X(\tau)|_{\mathrm{M}}\,|\Omega(\tau)|_{\mathrm{M}}\,d\tau\right| \\ &= \left|\int_{t_0}^t e^{-k\tau}\,|Y(\tau) - X(\tau)|_{\mathrm{M}}\,e^{k\tau}\,|\Omega(\tau)|_{\mathrm{M}}\,d\tau\right| \end{aligned}$$

$$\begin{aligned} &\leqq \|Y - X\|_{J,k} \sup_J |\Omega|_{\mathrm{M}} \left|\int_{t_0}^t e^{k\tau}\,d\tau\right| \\ &= \|Y - X\|_{J,k} \frac{\sup_J |\Omega|_{\mathrm{M}}}{|k|} e^{kt} \left|1 - e^{-k(t-t_0)}\right| \end{aligned}$$

holds for $t \in J$. Here, setting $J = [t_0, a]$ and $k = 2\sup_J |\Omega|_{\mathrm{M}}$, we have

$$\|Y - X\|_{J,k} \leqq \frac{1}{2}\|Y - X\|_{J,k},$$

that is, $\|Y - X\|_{J,k} = 0$, proving $Y(t) = X(t)$ for $t \in J$. Similarly, on the interval $J' = [a, t_0]$, we can conclude $Y = X$ on J' setting $k = -2\sup_J |\Omega|_{\mathrm{M}}$. Since J and J' are arbitrary, $Y = X$ holds on I.
Existence: Let $J := [t_0, a] \subset I$ be a closed interval, and define a sequence $\{X_j\}$ of matrix-valued functions defined on I satisfying $X_0(t) = \mathrm{id}$ and

$$X_{j+1}(t) = \mathrm{id} + \int_{t_0}^t X_j(\tau)\Omega(\tau)\,d\tau \quad (j = 0, 1, 2, \dots). \tag{1.7}$$

Let $k := 2\sup_J |\Omega|_{\mathrm{M}}$. Then

$$\begin{aligned} |X_{j+1}(t) - X_j(t)|_{\mathrm{M}} &\leqq \int_{t_0}^t |X_j(\tau) - X_{j-1}(\tau)|_{\mathrm{M}} |\Omega(\tau)|_{\mathrm{M}}\,d\tau \\ &\leqq e^{kt}\|X_j - X_{j-1}\|_{J,k} \frac{\sup_J |\Omega|_{\mathrm{M}}}{k} = \frac{e^{kt}}{2}\|X_j - X_{j-1}\|_{J,k}, \end{aligned}$$

and hence $\|X_{j+1} - X_j\|_{J,k} \leqq \frac{1}{2}\|X_j - X_{j-1}\|_{J,k}$, that is, $\{X_j\}$ is a Cauchy sequence with respect to $\|\cdot\|_{J,k}$. Thus, by completeness

(Lemma 1.5), it converges to some $X \in C^0(J, \mathrm{M}_n(\mathbb{R}))$. By (1.7), the limit X satisfies

$$X(t_0) = \mathrm{id}, \qquad X(t) = \mathrm{id} + \int_{t_0}^{t} X(\tau)\Omega(\tau)\,d\tau.$$

Applying the fundamental theorem of calculus, we can see that X satisfies $X'(t) = X(t)\Omega(t)$ ($' = d/dt$). Since J can be taken arbitrarily, existence of the solution on $I \cap \{t \geqq t_0\}$ is proved. Existence of $I \cap \{t \leqq t_0\}$ can be proved in the same way. So far, existence of a differentiable function $X(t)$ satisfying (1.6) is obtained.

Finally, we shall prove that X is of class C^∞. Since $X'(t) = X(t)\Omega(t)$, the derivative X' of X is continuous. Hence X is of class C^1, and so is $X(t)\Omega(t)$. Thus we have that $X'(t)$ is of class C^1, and then X is of class C^2. Iterating this argument, we can prove that $X(t)$ is of class C^r for arbitrary r. □

Corollary 1.7. *Let $\Omega(t)$ be a matrix-valued C^∞-function defined on an interval I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,X_0}(t)$ defined on I such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0. \tag{1.8}$$

In particular, $X_{t_0,X_0}(t)$ is of class C^∞ in X_0 and t.

Proof. We rewrite $X(t)$ in Proposition 1.6 as $Y(t) = X_{t_0,\mathrm{id}}(t)$. Then the function

$$X(t) := X_0 Y(t) = X_0 X_{t_0,\mathrm{id}}(t), \tag{1.9}$$

is desired one. Conversely, assume $X(t)$ satisfies the conclusion. Noticing $Y(t)$ is a regular matrix for all t because of Proposition 1.3,

$$W(t) := X(t)Y(t)^{-1}$$

satisfies

$$\begin{aligned}\frac{dW}{dt} &= \frac{dX}{dt}Y^{-1} - X(t)Y^{-1}\frac{dY}{dt}Y^{-1}\\ &= X\Omega Y^{-1} - XY^{-1}Y\Omega Y^{-1} = O.\end{aligned}$$

Hence

$$W(t) = W(t_0) = X(t_0)Y(t_0)^{-1} = X_0.$$

Hence the uniqueness is obtained. The final part is obvious by the expression (1.9). □

Proposition 1.8. *Let $\Omega(t)$ and $B(t)$ be a matrix-valued C^∞-functions defined on I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function defined on I satisfying*

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0. \tag{1.10}$$

Proof. Rewrite X in Proposition 1.6 as $Y(t) := X_{t_0,\mathrm{id}}(t)$. Then

$$X(t) = \left(X_0 + \int_{t_0}^{t} B(\tau)Y^{-1}(\tau)\,d\tau\right)Y(t) \tag{1.11}$$

satisfies (1.10). Conversely, if X satisfies (1.10), $W := XY^{-1}$ satisfies

$$X' = W'Y + WY' = W'Y + WY\Omega, \quad X\Omega + B = WY\Omega + B,$$

and then we have $W' = BY^{-1}$. Since $W(t_0) = X_0$,

$$W = X_0 + \int_{t_0}^{t} B(\tau)Y^{-1}(\tau)\,d\tau.$$

Thus we obtain (1.11). $\square$

Theorem 1.9. *Let I and U be an interval and a domain in $\mathbb{R}^m$, respectively, and let $\Omega(t,\boldsymbol{\alpha})$ and $B(t,\boldsymbol{\alpha})$ be matrix-valued C^∞-functions defined on $I \times U$ ($\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m)$). Then for each $t_0 \in I$, $\boldsymbol{\alpha} \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,X_0,\boldsymbol{\alpha}}(t)$ defined on I such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t,\boldsymbol{\alpha}) + B(t,\boldsymbol{\alpha}), \qquad X(t_0) = X_0. \tag{1.12}$$

Moreover,

$$I \times I \times \mathrm{M}_n(\mathbb{R}) \times U \ni (t,t_0,X_0,\boldsymbol{\alpha}) \mapsto X_{t_0,X_0,\boldsymbol{\alpha}}(t) \in \mathrm{M}_n(\mathbb{R})$$

is C^∞-map.

Proof. Let $\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) := \Omega(t+t_0,\boldsymbol{\alpha})$ and $\widetilde{B}(t,\tilde{\boldsymbol{\alpha}}) = B(t+t_0,\boldsymbol{\alpha})$, and let $\widetilde{X}(t) := X(t+t_0)$. Then (1.12) is equivalent to

$$\frac{d\widetilde{X}(t)}{dt} = \widetilde{X}(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) + \widetilde{B}(t,\tilde{\boldsymbol{\alpha}}), \quad \widetilde{X}(0) = X_0, \tag{1.13}$$

where $\tilde{\boldsymbol{\alpha}} := (t_0,\alpha_1,\dots,\alpha_m)$. There exists the unique solution $\widetilde{X}(t) = \widetilde{X}_{\mathrm{id},X_0,\tilde{\boldsymbol{\alpha}}}(t)$ of (1.13) for each $\tilde{\boldsymbol{\alpha}}$ because of Proposition 1.8. So it is sufficient to show differentiability with respect

to the parameter $\tilde{\boldsymbol{\alpha}}$. We set $Z = Z(t)$ as the unique solution of

$$\frac{dZ}{dt} = Z\widetilde{\Omega} + \widetilde{X}\frac{\partial\widetilde{\Omega}}{\partial\alpha_j} + \frac{\partial\widetilde{B}}{\partial\alpha_j}, \qquad Z(0) = O. \tag{1.14}$$

Then it holds that $Z = \partial\widetilde{X}/\partial\alpha_j$ (Problem 1-1). In particular, by the proof of Proposition 1.8, it holds that

$$Z = \frac{\partial\widetilde{X}}{\partial\alpha_j} = \left(\int_0^t \left(\widetilde{X}(\tau)\frac{\partial\widetilde{\Omega}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j} + \frac{\partial\widetilde{B}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j}\right)Y^{-1}(\tau)d\tau\right)Y(t).$$

Here, $Y(t)$ is the unique matrix-valued C^∞-function satisfying $Y'(t) = Y(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}})$, and $Y(0) = \mathrm{id}$. Hence $\widetilde{X}$ is a C^∞-function in $(t,\tilde{\boldsymbol{\alpha}})$. $\square$

Fundamental Theorem for Space Curves. As an application, we prove the fundamental theorem for space curves. A C^∞-map $\gamma\colon I \to \mathbb{R}^3$ defined on an interval $I \in \mathbb{R}$ into $\mathbb{R}^3$ is said to be a *regular curve* if $\dot{\gamma} \neq \mathbf{0}$ holds on I. For a regular curve $\gamma(t)$, there exists a parameter change $t = t(s)$ such that $\tilde{\gamma}(s) := \gamma(t(s))$ satisfies $|\tilde{\gamma}'(s)| = 1$. Such a parameter s is called the *arc-length parameter*.

Let $\gamma(s)$ be a regular curve in $\mathbb{R}^3$ parametrized by the arc-length satisfying $\gamma''(s) \neq \mathbf{0}$ for all s. Then

$$\boldsymbol{e}(s) := \gamma'(s), \qquad \boldsymbol{n}(s) := \frac{\gamma''(s)}{|\gamma''(s)|}, \qquad \boldsymbol{b}(s) := \boldsymbol{e}(\boldsymbol{s}) \times \boldsymbol{n}(s)$$

forms a positively oriented orthonormal basis $\{\boldsymbol{e},\boldsymbol{n},\boldsymbol{b}\}$ of $\mathbb{R}^3$ for each s. Regarding each vector as column vector, we have the matrix-valued function

$$\mathcal{F}(s) := (\boldsymbol{e}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)) \in \mathrm{SO}(3). \tag{1.15}$$

in s, which is called the *Frenet frame* associated to the curve γ. Under the situation above, we set

$$\kappa(s) := |\gamma''(s)| > 0, \qquad \tau(s) := -\left\langle \boldsymbol{b}'(s), \boldsymbol{n}(s)\right\rangle,$$

which is called the *curvature* and *torsion*, respectively, of γ. Using these quantities, the Frenet frame satisfies

$$\frac{d\mathcal{F}}{ds} = \mathcal{F}\Omega, \qquad \Omega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}. \tag{1.16}$$

Proposition 1.10. *The curvature and the torsion are invariant under the transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ of $\mathbb{R}^3$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$). Conversely, two curves $\gamma_1(s)$, $\gamma_2(s)$ parametrized by arc-length parameter have common curvature and torsion, there exist $A \in \mathrm{SO}(3)$ and $\boldsymbol{b} \in \mathbb{R}^3$ such that $\gamma_2 = A\gamma_1 + \boldsymbol{b}$.*

Proof. Let κ, τ and $\mathcal{F}_1$ be the curvature, torsion and the Frenet frame of γ_1, respectively. Then the Frenet frame of $\gamma_2 = A\gamma_1 + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) is $\mathcal{F}_2 = A\mathcal{F}_1$. Hence both $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy (1.16), and then γ_1 and γ_2 have common curvature and torsion.

Conversely, assume γ_1 andγ_2 have common curvature and torsion. Then the frenet frame $\mathcal{F}_1$, $\mathcal{F}_2$ both satisfy (1.16). Let $\mathcal{F}$ be the unique solution of (1.16) with $\mathcal{F}(t_0) = \mathrm{id}$. Then by the proof of Corollary 1.7, we have $\mathcal{F}_j(t) = \mathcal{F}_j(t_0)\mathcal{F}(t)$ ($j = 1, 2$). In particular, since $\mathcal{F}_j \in \mathrm{SO}(3)$, $\mathcal{F}_2(t) = A\mathcal{F}_1(t)$ ($A := \mathcal{F}_2(t_0)\mathcal{F}_1(t_0)^{-1} \in \mathrm{SO}(3)$). Comparing the first column of these, $\gamma_2'(s) = A\gamma_1'(t)$ holds. Integrating this, the conclusion follows. □

Theorem 1.11 (The fundamental theorem for space curves)**.** *For given C^∞-functions $\kappa(s)$ and $\tau(s)$ defined on I such that $\kappa(s) > 0$ on I. Then there exists a space curve $\gamma(s)$ parametrized by arc-length whose curvature and torsion are κ and τ, respectively. Moreover, such a curve is unique up to transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) of $\mathbb{R}^3$.*

Proof. We have already shown the uniqueness in Proposition 1.10. We shall prove the existence: Let $\Omega(s)$ be as in (1.16), and $\mathcal{F}(s)$ the solution of (1.16) with $\mathcal{F}(s_0) = \mathrm{id}$. Since Ω is skew-symmetric, $\mathcal{F}(s) \in \mathrm{SO}(3)$ by Proposition 1.4. Denoting the column vectors of $\mathcal{F}$ by $\boldsymbol{e}$, $\boldsymbol{n}$, $\boldsymbol{b}$, and let

$$\gamma(s) := \int_{s_0}^{s} \boldsymbol{e}(\sigma)\, d\sigma.$$

Then $\mathcal{F}$ is the frenet frame of γ, and κ, and τ are the curvature and torsion of γ, respectively (Problem 1-2). □

Exercises

1-1 Verify that Z in (1.14) coincides with $\partial\widetilde{X}/\partial\alpha_j$.

1-2 Complete the proof of Theorem 1.11.

Integrability Conditions

Let $\Omega(u,v)$ and $\Lambda(u,v)$ be $n\times n$-matrix valued C^∞-maps defined on a domain $U\subset\mathbb{R}^2$. In this section, we consider an initial value problem of a system of linear partial differential equations

$$\frac{\partial X}{\partial u}=X\Omega,\qquad \frac{\partial X}{\partial v}=X\Lambda,\qquad X(u_0,v_0)=X_0, \tag{2.1}$$

where $(u_0,v_0)\in U$ is a fixed point, X is an $n\times n$-matrix valued unknown, and $X_0\in \mathrm{M}_n(\mathbb{R})$.

Proposition 2.1. *If a matrix-valued C^∞-function $X(u,v)$ defined on $U\subset\mathbb{R}^2$ satisfies* (2.1) *with $X_0\in \mathrm{GL}(n,\mathbb{R})$, then $X(u,v)\in\mathrm{GL}(n,\mathbb{R})$ for all $(u,v)\in U$. In addition, if Ω and Λ are skew-symmetric and $X_0\in\mathrm{SO}(n)$, then $X\in\mathrm{SO}(n)$ holds on U.*

Proof. Take a smooth path $\gamma\colon[0,1]\to U$ joining (u_0,v_0) and (u,v), and write $\gamma(t)=(u(t),v(t))$[4]. Setting $\widetilde{X}(t):=X\circ\gamma(t)=X\bigl(u(t),v(t)\bigr)$, (2.1) implies

$$\frac{d\widetilde{X}}{dt}=\widetilde{X}\left(\frac{du}{dt}\Omega+\frac{dv}{dt}\Lambda\right),\qquad \widetilde{X}(0)=X_0.$$

Hence, by Proposition 1.3, $\det\widetilde{X}(1)\neq 0$. The latter half of the statement follows from Proposition 1.4. □

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[4]Since U is connected, there exists a *continuous* path $\gamma\colon[0,1]\to U$ joining (u_0,v_0) and (u,v). Then one can find a smooth curve $\tilde\gamma$ joining these points as follows: For each $t\in[0,1]$, there exists a positive number $\rho_t>0$ such that $B_{\rho_t}(\gamma(t))\subset U$. Since $\gamma([0,1])$ is compact, there exists a finite sequence $0=t_0<t_1<\dots<t_N=1$ such that $\gamma([0,1])=\cup_{j=0}^N B_{\rho_{t_j}}(\gamma(t_j))$, where $B_\varepsilon(p)$ denotes a disk of radius ε centered at p. Choose $p_j\in B_{\rho_{t_{j-1}}}(\gamma(t_{j-1}))\cap B_{\rho_{t_j}}(\gamma(t_j))$ $(j=1,\dots,N)$. Then the polygonal line with vertices $\{\gamma(0),p_1,\dots,p_N,\gamma(1)\}$ lies on U and a piecewise linear path joining $\gamma(0)=(u_0,v_0)$ and $\gamma(1)=(u,v)$. Modifying such a path at vertices, we have a smooth path joining $\gamma(0)$ and $\gamma(1)$ (cf. see [2-1, Appendix B-5]).

Lemma 2.2. *If a matrix-valued C^∞ function $X\colon U\to\mathrm{GL}(n,\mathbb{R})$ satisfies* (2.1)*, it holds that*

$$\Omega_v-\Lambda_u=\Omega\Lambda-\Lambda\Omega. \tag{2.2}$$

Proof. Differentiating the first (resp. second) equation of (2.1) by v (resp. u), we have

$$X_{uv}=X_v\Omega+X\Omega_v=X(\Lambda\Omega+\Omega_v),$$
$$X_{vu}=X_u\Lambda+X\Lambda_u=X(\Omega\Lambda+\Lambda_u).$$

These two matrices coincide Since X is of class C^∞. Hence we have the conclusion. □

The equality (2.2) is called the *integrability condition* or *compatibility condition* of (2.1).

Frobenius' theorem In this section, we shall prove the following

Theorem 2.3. *Let $\Omega(u,v)$ and $\Lambda(u,v)$ be $n\times n$-matrix valued C^∞-functions defined on a simply connected domain $U\subset\mathbb{R}^2$*

satisfying (2.2). *Then for each* $(u_0, v_0) \in U$ *and* $X_0 \in \mathrm{M}_n(\mathbb{R})$, *there exists the unique* $n \times n$*-matrix valued function* $X \colon U \to \mathrm{M}_n(\mathbb{R})$ (2.1). *Moreover,*

- *if* $X_0 \in \mathrm{GL}(n,\mathbb{R})$, $X(u,v) \in \mathrm{GL}(n,\mathbb{R})$ *holds on* U,
- *if* $\operatorname{tr} \Omega = \operatorname{tr} \Lambda = 0$ *holds on* U *and* $X_0 \in \mathrm{SL}(n,\mathbb{R})$, $X(u,v) \in \mathrm{SL}(n,\mathbb{R})$ *holds on* U,
- *if* Ω *and* Λ *are skew-symmetric matrices, and* $X_0 \in \mathrm{SO}(n)$, $X(u,v) \in \mathrm{SO}(n)$ *holds on* U.

To prove Theorem 2.3, it is sufficient to show for the case $U = \mathbb{R}^2$. In fact, by Lemma 2.4 and Fact 2.5 below, we can replace U with $\mathbb{R}^2$ by an appropriate coordinate change.

Lemma 2.4. *Let* $V \ni (\xi,\eta) \mapsto (u,v) \in U$ *be a diffeomorphism between domains* V, $U \subset \mathbb{R}^2$, *and let* $\Omega = \Omega(u,v)$ *and* $\Lambda = \Lambda(u,v)$ *be matrix-valued functions on* U. *Set*

$$
\begin{aligned}
\widetilde{\Omega}(\xi,\eta) &:= \Omega\bigl(u(\xi,\eta), v(\xi,\eta)\bigr)\frac{\partial u}{\partial \xi} + \Lambda\bigl(u(\xi,\eta), v(\xi,\eta)\bigr)\frac{\partial v}{\partial \xi},\\
\widetilde{\Lambda}(\xi,\eta) &:= \Omega\bigl(u(\xi,\eta), v(\xi,\eta)\bigr)\frac{\partial u}{\partial \eta} + \Lambda\bigl(u(\xi,\eta), v(\xi,\eta)\bigr)\frac{\partial v}{\partial \eta}.
\end{aligned}
\tag{2.3}
$$

If a matrix-valued function $X \colon U \to \mathrm{M}_n(\mathbb{R})$ *satisfies* (2.1), $\widetilde{X}(\xi,\eta) = X(u(\xi,\eta), v(\xi,\eta))$ *satisfies*

$$
\frac{\partial \widetilde{X}}{\partial \xi} = \widetilde{X}\widetilde{\Omega}, \quad \frac{\partial \widetilde{X}}{\partial \eta} = \widetilde{X}\widetilde{\Lambda}, \quad \widetilde{X}(\xi_0,\eta_0) = X_0, \tag{2.4}
$$

where $\bigl(u(\xi_0,\eta_0), v(\xi_0,\eta_0)\bigr) = (u_0, v_0)$. *Moreover, the integrability condition* (2.2) *of* (2.1) *is equivalent to that of* (2.4).

Proof. The equation (2.1) can be considered as a equality of 1-forms

$$
dX = X\Theta, \qquad \Theta := \Omega\,du + \Lambda\,dv,
$$

which does not depend on a choice of coordinate systems. If we write

$$
\Theta = \Omega\,du + \Lambda\,dv = \widetilde{\Omega}\,d\xi + \widetilde{\Lambda}\,d\eta,
$$

Ω, Λ, $\widetilde{\Omega}$ and $\widetilde{\Lambda}$ satisfy (2.3). Here, the integrability condition can be rewritten as

$$
d\Theta + \Theta \wedge \Theta = O,
$$

which is an equality of 2-forms. This does not depend on coordinates, the conclusion follows. □

Fact 2.5. *A simply connected domain in* $\mathbb{R}^2$ *is diffeomorphic to* $\mathbb{R}^2$.

In fact, the Riemann mapping theorem yields the fact above[5].

Proof of Theorem 2.3. By Lemma 2.4 and Fact 2.5, we may assume $U = \mathbb{R}^2$, $(u_0, v_0) = (0,0)$ without loss of generality.

Existence: By the fundamental theorem of linear ordinary differential equations (Corollary 1.7), there exists the unique C^∞-map $F \colon \mathbb{R} \to \mathrm{M}_n(\mathbb{R})$ such that

$$
\frac{dF}{du}(u) = F(u)\Omega(u,0) \qquad F(0) = X_0.
$$

[5]Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, a simply connected domain of $U = \mathbb{R}^2$ is conformally equivalent to the unit disc $D := \{z \in \mathbb{C} \,|\, |z| < 1\}$ or $\mathbb{C}$, because of the Riemann mapping theorem (cf. [2-3]). Though D and $\mathbb{C}$ are not conformally equivalent, D and $\mathbb{R}^2$ are diffeomorphic. Then any simply connected domain is diffeomorphic to $\mathbb{R}^2$.

For each $u \in \mathbb{R}$, we denote by $G^u(v)$ the unique solution of the ordinary differential equation

$$\frac{dG^u}{dv}(v) = G^u(v)\Lambda(u,v), \qquad G^u(0) = F(u)$$

in v. Then the function $X(u,v) := G^u(v)$ is the desired one. In fact, the solution of a ordinary differential equation depends smoothly on the initial value, $X(u,v)$ is a matrix-valued C^∞ function defined on $\mathbb{R}^2$. By definition of $G^u(v)$, we have

$$\frac{\partial X}{\partial v}(u,v) = \frac{dG^u}{dv}(v) = G^u(v)\Lambda(u,v) = X(u,v)\Lambda(u,v). \tag{2.5}$$

Since X is C^∞, $X_{uv} = X_{vu}$ holds. Then by the integrability condition (2.2), it holds that

$$\begin{aligned}
\frac{\partial}{\partial v}\left(\frac{\partial X}{\partial u} - X\Omega\right) &= \frac{\partial}{\partial u}\frac{\partial X}{\partial v} - \frac{\partial X}{\partial v}\Omega - X\frac{\partial \Omega}{\partial v} \\
&= \frac{\partial}{\partial u}(X\Lambda) - \frac{\partial X}{\partial v}\Omega - X\frac{\partial \Omega}{\partial v} \\
&= \frac{\partial X}{\partial u}\Lambda + X\frac{\partial \Lambda}{\partial u} - \frac{\partial X}{\partial v}\Omega - X\frac{\partial \Omega}{\partial v} \\
&= X(\Lambda_u - \Omega_v) + \frac{\partial X}{\partial u}\Lambda - \frac{\partial X}{\partial v}\Omega \\
&= X(\Lambda_u - \Omega_v - \Lambda\Omega) + \frac{\partial X}{\partial u}\Lambda \\
&= -X\Omega\Lambda + \frac{\partial X}{\partial u}\Lambda \\
&= \left(\frac{\partial X}{\partial u} - X\Omega\right)\Lambda.
\end{aligned}$$

That is, for each fixed u, the map $H(v) := X_u(u,v) - X\Omega$ satisfies an ordinary differential equation in v as follows:

$$\frac{dH}{dv}(u,v) = H(u,v)\Lambda(u,v).$$

Letting $v = 0$, we have

$$\begin{aligned}
H(u,0) &= X_u(u,0) - X(u,0)\Omega(u,0) \\
&= (G^u)_u(u,0) - G^u(0)\Omega(u,0) \\
&= F'(u) - F(u)\Omega(u,0) = O
\end{aligned}$$

and then, by uniqueness of the solutions of initial value problems for ordinary differential equations, $H(u,v) = 0$ holds. Since (u,v) is arbitrarily taken, we have

$$\frac{\partial X}{\partial u}(u,v) = X(u,v)\Omega(u,v),$$

that is, $X(u,v)$ is the solution of (2.1).

Uniqueness: Let X and $\hat{X}$ be matrix-valued functions satisfying (2.1). Then $\hat{X} - X$ is a solution of (2.1) with $X_0 = O$ since (2.1) is linear. Hence, to show the uniqueness, it is sufficient to show that the solution X of (2.1) with initial condition $X_0 = O$ is the constant function $X(u,v) = O$.

Let X be such a solution of (2.1). Here, $X(0,0) = O$ as we have set $(u_0, v_0) = (0,0)$. For an arbitrary $(u,v) \in \mathbb{R}^2$, let $F(t) := X(tu, tv)$. Then

$$\begin{aligned}
\frac{d}{dt}F(t) &= uX_u(tu,tv) + vX_v(tu,tv) \\
&= X(tu,tv)(u\Omega(tu,tv) + v\Lambda(tu,tv)) = F(t)\omega(t)
\end{aligned} \tag{2.6}$$

holds, where $\omega(t) = u\Omega(tu,tv) + v\Lambda(tu,tv)$. Then the ordinary differential equation (2.6) for $F(t)$ in t, the uniqueness of solutions of ordinary differential equations yields $F(t) = O$ since $F(0) = X(0,0) = O$. In particular, we have $X(u,v) = F(1) = O$. Since (u,v) has been taken arbitrarily, $X(u,v) = 0$ holds for all $(u,v) \in \mathbb{R}^2$. Hence we have the uniqueness. □

Application: Poincaré's lemma.

Theorem 2.6 (Poincaré's lemma)**.** *If a differential* 1*-form*

$$\omega = \alpha(u,v)\,du + \beta(u,v)\,dv$$

defined on a simply connected domain $U \subset \mathbb{R}^2$ is closed, that is, $d\omega = 0$ holds, then there exists a C^∞-function f on U such that $df = \omega$. Such a function f is unique up to additive constants.

Proof. Since $d\omega = (\beta_u - \alpha_v)\,du \wedge dv$, the assumption is equivalent to

$$\beta_u - \alpha_v = 0. \tag{2.7}$$

Consider a system of linear partial differential equations with unknown a 1×1-matrix valued function (i.e. a real-valued function) $\xi(u,v)$ as

$$\frac{\partial \xi}{\partial u} = \xi\alpha, \qquad \frac{\partial \xi}{\partial v} = \xi\beta, \qquad \xi(u_0,v_0) = 1. \tag{2.8}$$

Then it satisfies (2.2) because of (2.7). Hence by Theorem 2.3, there exists a smooth function $\xi(u,v)$ satisfying (2.8). In particular, Proposition 1.3 yields $\xi = \det \xi$ never vanishes. Since

$\xi(u_0,v_0) = 1 > 0$, this means that $\xi > 0$ holds on U. Letting $f := \log \xi$, we have the function f satisfying $df = \omega$.

Next, we show the uniqueness: if two functions f and g satisfy $df = dg = \omega$, it holds that $d(f-g) = 0$. Hence by connectivity of U, $f - g$ must be constant. □

Application: Conjugation of Harmonic functions. In this paragraph, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. It is well-known that a function

$$f\colon U \ni u + \mathrm{i}v \longmapsto \xi(u,v) + \mathrm{i}\eta(u,v) \in \mathbb{C} \qquad (\mathrm{i} = \sqrt{-1}) \tag{2.9}$$

defined on a domain $U \subset \mathbb{C}$ is *holomorphic* if and only if it satisfies the following relation, called the *Cauchy-Riemann equations*:

$$\frac{\partial \xi}{\partial u} = \frac{\partial \eta}{\partial v}, \qquad \frac{\partial \xi}{\partial v} = -\frac{\partial \eta}{\partial u}. \tag{2.10}$$

Definition 2.7. A function $f\colon U \to \mathbb{R}$ defined on a domain $U \subset \mathbb{R}^2$ is said to be *harmonic* if it satisfies

$$\Delta f = f_{uu} + f_{vv} = 0.$$

The operator Δ is called the *Laplacian*.

Proposition 2.8. *If function f in* (2.9) *is holomorphic, $\xi(u,v)$ and $\eta(u,v)$ are harmonic functions.*

Proof. By (2.10), we have

$$\xi_{uu} = (\xi_u)_u = (\eta_v)_u = \eta_{vu} = \eta_{uv} = (\eta_u)_v = (-\xi_v)_v = -\xi_{vv}.$$

Hence $\Delta\xi = 0$. Similarly,

$$\eta_{uu} = (-\xi_v)_u = -\xi_{vu} = -\xi_{uv} = -(\xi_u)_v = -(\eta_v)_v = -\eta_{vv}.$$

Thus $\Delta\eta = 0$. □

Theorem 2.9. *Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u,v)$ a C^∞-function harmonic on U*[6]*. Then there exists a C^∞ harmonic function η on U such that $\xi(u,v) + \mathrm{i}\eta(u,v)$ is holomorphic on U.*

Proof. Let $\alpha := -\xi_v\,du + \xi_u\,dv$. Then by the assumption,

$$d\alpha = (\xi_{vv} + \xi_{uu})\,du \wedge dv = 0$$

holds, that is, α is a closed 1-form. Hence by simple connectivity of U and the Poincaré's lemma (Theorem 2.6), there exists a function η such that $d\eta = \eta_u\,du + \eta_v\,dv = \alpha$. Such a function η satisfies (2.10) for given ξ. Hence $\xi + \mathrm{i}\eta$ is holomorphic in $u + \mathrm{i}v$. □

Example 2.10. A function $\xi(u,v) = e^u \cos v$ is harmonic. Set

$$\alpha := -\xi_v\,du + \xi_u\,dv = e^u \sin v\,du + e^u \cos v\,dv.$$

Then $\eta(u,v) = e^u \sin v$ satisfies $d\eta = \alpha$. Hence

$$\xi + \mathrm{i}\eta = e^u(\cos v + \mathrm{i}\sin v) = e^{u+\mathrm{i}v}$$

is holomorphic in $u + \mathrm{i}v$.

Definition 2.11. The harmonic function η in Theorem 2.9 is called the *conjugate* harmonic function of ξ.

[6] The theorem holds under the assumption of C^2-differentiablity.

The fundamental theorem for Surfaces. Let $p\colon U \to \mathbb{R}^3$ be a parametrization of a *regular surface* defined on a domain $U \subset \mathbb{R}^2$. That is, $p = p(u,v)$ is a C^∞-map such that p_u and p_v are linearly independent at each point on U. Then $\nu := (p_u \times p_v)/|p_u \times p_v|$ is the *unit normal vector field* to the surface. The matrix-valued function $\mathcal{F} := (p_u, p_v, \nu)\colon U \to \mathrm{M}_3(\mathbb{R})$ is called the *Gauss frame* of p. We set

$$\begin{aligned} ds^2 &:= E\,du^2 + 2F\,du\,dv + G\,dv^2, \\ I\!I &:= L\,du^2 + 2M\,du\,dv + N\,dv^2, \end{aligned} \tag{2.11}$$

where

$$\begin{array}{lll} E = p_u \cdot p_u & F = p_u \cdot p_v & G = p_v \cdot p_v \\ L = p_{uu} \cdot \nu & M = p_{uv} \cdot \nu & N = p_{vv} \cdot \nu. \end{array}$$

We call ds^2 (resp. $I\!I$) the *first* (resp. *second*) *fundamental form.* Note that linear independence of p_u and p_v implies

$$E > 0, \qquad G > 0 \qquad \text{and} \qquad EG - F^2 > 0. \tag{2.12}$$

Set

$$\begin{aligned} \varGamma^1_{11} &:= \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, \\ \varGamma^2_{11} &:= \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}, \\ \varGamma^1_{12} &= \varGamma^1_{21} := \frac{GE_v - FG_u}{2(EG - F^2)}, \end{aligned} \tag{2.13}$$

$$
\begin{aligned}
\mathit{\Gamma}_{12}^2 &= \mathit{\Gamma}_{21}^2 := \frac{EG_u - FE_v}{2(EG - F^2)},\\
\mathit{\Gamma}_{22}^1 &:= \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)},\\
\mathit{\Gamma}_{22}^2 &:= \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.
\end{aligned}
$$

and

$$
A = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix} := \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}. \tag{2.14}
$$

The functions $\mathit{\Gamma}_{ij}^k$ and the matrix A are called the *Christoffel symbols* and the *Weingarten matrix*. We state the following the *fundamental theorem for surfaces*, and give a proof (for a special case) in the following section.

Theorem 2.12 (The Fundamental Theorem for Surfaces)**.** *Let $p\colon U \ni (u,v) \mapsto p(u,v) \in \mathbb{R}^3$ be a parametrization of a regular surface defined on a domain $U \subset \mathbb{R}^2$. Then the Gauss frame $\mathcal{F} := \{p_u, p_v, \nu\}$ satisfies the equations*

$$
\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\mathit{\Omega}, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\mathit{\Lambda}, \tag{2.15}
$$

$$
\mathit{\Omega} := \begin{pmatrix} \mathit{\Gamma}_{11}^1 & \mathit{\Gamma}_{12}^1 & -A_1^1 \\ \mathit{\Gamma}_{11}^2 & \mathit{\Gamma}_{12}^2 & -A_1^2 \\ L & M & 0 \end{pmatrix}, \quad \mathit{\Lambda} := \begin{pmatrix} \mathit{\Gamma}_{21}^1 & \mathit{\Gamma}_{22}^1 & -A_2^1 \\ \mathit{\Gamma}_{21}^2 & \mathit{\Gamma}_{22}^2 & -A_2^2 \\ M & N & 0 \end{pmatrix},
$$

where $\mathit{\Gamma}_{jk}^i$ $(i,j,k = 1,2)$, A_l^k and L, M, N are the Christoffel symbols, the entries of the Weingarten matrix and the entries of the second fundamental form, respectively.

Theorem 2.13. *Let $U \subset \mathbb{R}^2$ be a simply connected domain, E, F, G, L, M, N C^∞-functions satisfying* (2.12)*, and $\mathit{\Gamma}_{ij}^k$, A_i^j the functions defined by* (2.13) *and* (2.14)*, respectively. If $\mathit{\Omega}$ and $\mathit{\Lambda}$ satisfies*

$$
\mathit{\Omega}_v - \mathit{\Lambda}_u = \mathit{\Omega}\mathit{\Lambda} - \mathit{\Lambda}\mathit{\Omega},
$$

there exists a parameterization $p\colon U \to \mathbb{R}^3$ of regular surface whose fundamental forms are given by (2.11)*. Moreover, such a surface is unique up to orientation preserving isometries of $\mathbb{R}^3$.*

References

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Exercises

2-1 Let $\xi(u,v) = \log\sqrt{u^2 + v^2}$ be a function defined on $U = \mathbb{R}^2 \setminus \{(0,0)\}$

(1) Show that ξ is harmonic on U.

(2) Find the conjugate harmonic function η of ξ on

$$
V = \mathbb{R}^2 \setminus \{(u,0) \mid u \leqq 0\} \subset U.
$$

(3) Show that there exists no conjugate harmonic function of ξ defined on U.

Isothermal parameters

A Review of Complex Analysis. Let $\mathbb{C}$ be the complex plane. A C^1-function[7] $f\colon \mathbb{C} \ni D \in z \mapsto w = f(z) \in \mathbb{C}$ defined on a domain D is said to be *holomorphic* if the derivative

$$f'(z) := \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists for all $z \in D$.

Fact 3.1 (The Cauchy-Riemann equation)**.** *A function $f\colon \mathbb{C} \ni D \to \mathbb{C}$ is holomorphic if and only if*

$$\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta} \qquad \text{and} \qquad \frac{\partial u}{\partial \eta} = -\frac{\partial v}{\partial \xi} \tag{3.1}$$

holds on D, where $w = f(z)$, $z = \xi + i\eta$, $w = u + iv$ ($i = \sqrt{-1}$).

For functions of complex variable $z = \xi + i\eta$, we set

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \eta}\right), \quad \frac{\partial}{\partial \bar z} := \frac{1}{2}\left(\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial \eta}\right). \tag{3.2}$$

Corollary 3.2. *For a complex function f,* (3.1) *is equivalent to*

$$\frac{\partial f}{\partial \bar z} = 0. \tag{3.3}$$

Proof. Setting $w = f(z) = u + iv$ and $z = \xi + i\eta$. Then the real (resp. imaginary) part of the left-hand side of (3.3) coincides with the first (resp. second) equation of (3.1). $\square$

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[7] Of class C^1 as a map from $D \subset \mathbb{R}^2$ to $\mathbb{R}^2$.

Isothermal Coordinates.

Definition 3.3. Let $f\colon M^2 \to \mathbb{R}^3$ be an immersion of 2-manifold, and ds^2 its first fundamental form. A local coordinate chart $\bigl(U;(u,v)\bigr)$ of M^2 is called an *isothermal coordinate system* or a *conformal coordinate system* if ds^2 is written in the form[8]

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad \sigma = \sigma(u,v) \in C^\infty(U).$$

Example 3.4. Let $\gamma(u) = (x(u), z(u)) = (a\cosh\frac{u}{a}, u)$, that is, γ is the graph $x = a\cosh\frac{z}{a}$ on the xz-plane, called the *catenary*. We call the surface of revolution generated by $\gamma(u)$ the *catenoid*, which is parametrized as

$$p(u,v) = \bigl(x(u)\cos v, x(u)\sin v, z(u)\bigr),$$

This parametrization of the catenoid is isothermal when $a = 1$. In fact, the first fundamental form is expressed as $\cosh^2(u/a)(du^2 + a^2dv^2)$.

Definition 3.5. Two charts $\bigl(U_j;(u_j,v_j)\bigr)$ ($j = 1,2$) of a 2-manifold M^2 has the *same* (resp. *opposite*) *orientation* if the Jacobian $\frac{\partial(u_2,v_2)}{\partial(u_1,v_1)}$ is positive (resp. negative) on $U_1 \cap U_2$. A manifold M^2 is said to be *oriented* if there exists an atlas $\bigl\{\bigl(U_j;(u_j,v_j)\bigr)\bigr\}$ such that all charts have the same orientation. A choice of such an atlas is called an *orientation* of M^2.

[8] The notion of the isothermal coordinate system can be defined not only for surfaces but also for Riemannian 2-manifolds, that is, differentiable 2-manifolds M^2 with Riemannian metrics ds^2 (the first fundamental forms).

Proposition 3.6. *Let (u,v) be an isothermal coordinate system of a surface. Then another coordinate system (ξ,η) is also isothermal if and only if the parameter change $(\xi,\eta)\mapsto(u,v)$ satisfy*

$$\frac{\partial u}{\partial \xi}=\varepsilon\frac{\partial v}{\partial \eta},\qquad \frac{\partial u}{\partial \eta}=-\varepsilon\frac{\partial v}{\partial \xi}, \tag{3.4}$$

where $\varepsilon=1$ (resp. -1) if (u,v) and (ξ,η) has the same (resp. the opposite) orientation.

Proof. If we write $ds^2=e^{2\sigma}(du^2+dv^2)$, it holds that

$$ds^2=e^{2\sigma}\bigl((u_\xi^2+v_\xi^2)\,d\xi^2+2(u_\xi u_\eta+v_\xi v_\eta)\,d\xi\,d\eta+(u_\eta^2+v_\eta^2)\,d\eta^2\bigr).$$

Thus, (ξ,η) is isothermal if and only if

$$u_\xi^2+v_\xi^2=u_\eta^2+v_\eta^2,\qquad u_\xi u_\eta+v_\xi v_\eta=0. \tag{3.5}$$

The second equality yields $(u_\eta,v_\eta)=\varepsilon(-v_\xi,u_\xi)$ for some function ε. Substituting this into the first equation of (3.5), we get $\varepsilon=\pm1$. Moreover,

$$\frac{\partial(u,v)}{\partial(\xi,\eta)}=\det\begin{pmatrix}u_\xi & u_\eta\\ v_\xi & v_\eta\end{pmatrix}=\det\begin{pmatrix}u_\xi & -\varepsilon v_\xi\\ v_\xi & \varepsilon u_\xi\end{pmatrix}=\varepsilon(u_\xi^2+u_\eta^2).$$

Thus, the conclusion follows. □

Corollary 3.7. *Let (u,v) is an isothermal coordinate system. Then a coordinate system (ξ,η) is isothermal and has the same orientation as (u,v) if and only if the map $\xi+i\eta\mapsto u+iv$ $(i=\sqrt{-1})$ is holomorphic.*

Proof. Equations (3.4) for $\varepsilon=+1$ are nothing but the Cauchy-Riemann equations (3.1). □

The notion of isothermal coordinate systems are meaningful not only for immersed surfaces but also for Riemannian manifolds. There exist such coordinate systems on a 2-dimensional Riemannian manifold:

Fact 3.8 (Section 15 in 3-1)**.** *Let (M^2,ds^2) be an arbitrary Riemannian manifold. Then for each $p\in M^2$, there exists an isothermal chart containing p.*

Corollary 3.9. *Any oriented Riemannian 2-manifold has a structure of Riemann surface (i.e., a complex 1-manifold) such that for each complex coordinate $z=u+iv$, (u,v) is an isothermal coordinate system for the Riemannian metric.*

Proof. Let $p\in M^2$ and take a local coordinate chart $\bigl(U_p;(x,y)\bigr)$ at p which is compatible to the orientation of M^2. Then there exists an isothermal coordinate chart $\bigl(V_p;(u_p,v_p)\bigr)$ at p, because of Fact 3.8. Moreover, replacing (u,v) by (v,u) if necessary, we can take (u,v) which has the same orientation of (x,y). Thus, we have an atlas $\bigl\{\bigl(V_p;(u_p,v_p)\bigr)\bigr\}$ consisting of isothermal coordinate systems. Since each chart is compatible to the orientation, the coordinate change $z_p=u_p+iv_p\mapsto u_q+iv_q=z_q$ is holomorphic. Hence we get a complex atlas $\bigl\{\bigl(V_p;z_p\bigr)\bigr\}$. □

The Gauss and Weingarten formulas. Let $p\colon U\to\mathbb{R}^3$ be a parametrized regular surface defined on a domain U of the uv-plane. Assume that (u,v) is an isothermal coordinate system, and write the first fundamental form ds^2 as

$$ds^2 := e^{2\sigma}(du^2 + dv^2) \qquad \sigma \in C^\infty(U), \tag{3.6}$$

that is,

$$p_u \cdot p_u = p_v \cdot p_v = e^{2\sigma}, \qquad p_u \cdot p_v = 0, \tag{3.7}$$

where "$\cdot$" denotes the canonical inner product of $\mathbb{R}^3$. Since

$$|p_u \times p_v| = \sqrt{(p_u \cdot p_u)(p_v \cdot p_v) - (p_u \cdot p_v)^2} = e^{2\sigma},$$

the unit normal vector field ν can be chosen as

$$\nu = e^{-2\sigma}(p_u \times p_v), \tag{3.8}$$

where "$\times$" denotes the vector product of $\mathbb{R}^3$. Write the second fundamental form of p as

$$II = L\,du^2 + 2M\,du\,dv + N\,dv^2, \tag{3.9}$$

where

$$L = p_{uu} \cdot \nu, \qquad M = p_{uv} \cdot \nu, \qquad N = p_{vv} \cdot \nu.$$

Proposition 3.10 (The Gauss formula)**.** *Under the situation above, it holds that*

$$\begin{aligned} p_{uu} &= \sigma_u p_u - \sigma_v p_v + L\nu, \\ p_{uv} &= \sigma_v p_u + \sigma_u p_v + M\nu, \\ p_{vv} &= -\sigma_u p_u + \sigma_v p_v + N\nu. \end{aligned}$$

Proof. Since $\{p_u, p_v, \nu\}$ is a basis of $\mathbb{R}^3$ for each $(u,v) \in U$, one can write

$$p_{uu} = a p_u + b p_v + c\nu, \tag{3.10}$$

where a, b, c are smooth functions on U. Here, since ν is a unit vector perpendicular to both p_u and p_v, we have

$$c = p_{uu} \cdot \nu = L.$$

On the other hand, by (3.7), we have

$$\begin{aligned} &e^{2\sigma} a = p_{uu} \cdot p_u = \frac{1}{2}(p_u \cdot p_u)_u = \frac{1}{2}(e^{2\sigma})_u = \sigma_u e^{2\sigma}, \\ &e^{2\sigma} b = p_{uu} \cdot p_v = (p_u \cdot p_v)_u - p_u \cdot p_{uv} = -\tfrac{1}{2}(p_u \cdot p_u)_v = -\sigma_v e^{2\sigma}. \end{aligned}$$

Thus the first equality of the conclusion is obtained. The second and third equality can be obtained in the same manner. □

Proposition 3.11 (The Weingarten formula)**.** *Under the situation above, it holds that*

$$\nu_u = -e^{-2\sigma}(L p_u + M p_v), \qquad \nu_v = -e^{-2\sigma}(M p_u + N p_v).$$

Proof. If we write $\nu_u = a p_u + b p_v + c\nu$, we have

$$\begin{aligned} e^{2\sigma} a &= \nu_u \cdot p_u = (\nu \cdot p_u)_u - \nu \cdot p_{uu} = -L, \\ e^{2\sigma} b &= \nu_u \cdot p_v = (\nu \cdot p_v)_u - \nu \cdot p_{uv} = -M, \\ c &= \nu_u \cdot \nu = \frac{1}{2}(\nu \cdot \nu)_u, \end{aligned}$$

and the first equality of the conclusion is obtained. The second equality can be proven in the same manner. □

Gauss Frame. As seen in the proofs of Proposition 3.10 and 3.11, $\{p_u, p_v, \nu\}$ is a basis of $\mathbb{R}^3$ for each $(u,v) \in U$. Regarding p_u, p_v and ν as column vectors, we then have a matrix-valued function

$$\mathcal{F} := (p_u, p_v, \nu)\colon U \longmapsto \mathrm{GL}(3,\mathbb{R}) \subset \mathrm{M}_3(\mathbb{R}). \tag{3.11}$$

We call such an $\mathcal{F}$ the *Gauss frame* of the surface. The following theorem is an immediate consequence of Propositions 3.10 and 3.11:

Theorem 3.12. *Let $p\colon U \to \mathbb{R}^3$ be a regular surface defined on a domain U in the uv-plane, and denote by ν the unit normal vector field of it. Assume that (u,v) is an isothermal coordinate system, and the first and second fundamental forms are written as*

$$ds^2 = e^{2\sigma}(du^2+dv^2), \qquad I\!I = L\,du^2 + 2M\,du\,dv + N\,dv^2. \tag{3.12}$$

Then the Gauss frame $\mathcal{F} := (p_u, p_v, \nu)$ satisfies the following system of linear partial differential equations:

$$\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda, \tag{3.13}$$

$$\Omega := \begin{pmatrix} \sigma_u & \sigma_v & -e^{-2\sigma}L \\ -\sigma_v & \sigma_u & -e^{-2\sigma}M \\ L & M & 0 \end{pmatrix}, \qquad \Lambda := \begin{pmatrix} \sigma_v & -\sigma_u & -e^{-2\sigma}M \\ \sigma_u & \sigma_v & -e^{-2\sigma}N \\ M & N & 0 \end{pmatrix},$$

Gauss-Codazzi equations. The coefficients Ω and Λ in (3.13) must satisfy the integrability condition (2.2) in Lemma 2.2.

Lemma 3.13. *The matrices Ω and Λ in* (3.13) *satisfy*

$$\Omega_v - \Lambda_u - \Omega\Lambda + \Lambda\Omega = O$$

if and only if

$$\sigma_{uu} + \sigma_{vv} + e^{-2\sigma}(LN - M^2) = 0 \tag{3.14}$$

and

$$L_v - M_u = \sigma_v(L+N) \qquad \text{and} \qquad N_u - M_v = \sigma_u(L+N). \tag{3.15}$$

Proof. A direct computation. □

Thus we have

Theorem 3.14 (The Gauss and Codazzi equatoins)**.** *Let $p\colon U \to \mathbb{R}^3$ be a regular surface defined on a domain U in the uv-plane, and denote by ν the unit normal vector field of it. Assume that (u,v) is an isothermal coordinate system, and the first and second fundamental forms are written as* (3.12)*. Then* (3.14) *and* (3.15) *hold.*

Remark 3.15. The equations (3.14) and (3.15) are called the *Gauss equation* and the *Codazzi equations*, respectively. The Gauss equation is often referred as *Gauss' Theorema Egregium.*

Fundamental Theorem for Surfaces. The following is the special case of the fundamental theorem for surfaces (Theorem 2.13):

Theorem 3.16. *Let $U \subset \mathbb{R}^2$ be a simply connected domain, and let σ, L, M, N be C^∞-functions satisfying (3.14) and (3.15). Then there exists a parametrization $p\colon U \to \mathbb{R}^3$ of regular surface whose fundamental forms are given by (3.12). Moreover, such a surface is unique up to orientation preserving isometries of $\mathbb{R}^3$.*

Proof. By Lemma 3.13, Theorem 2.3 yields that there exists a matrix-valued function $\mathcal{F}\colon U \to \mathrm{M}_3(\mathbb{R})$ satisfying (3.13) with the initial condition

$$\mathcal{F}(u_0,v_0) = \begin{pmatrix} e^{\sigma(u_0,v_0)} & 0 & 0 \\ 0 & e^{\sigma(u_0,v_0)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.16}$$

for a fixed point $(u_0,v_0) \in U$. Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be vector-valued functions such that $\mathcal{F} = (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$. Since

$$\boldsymbol{a}_v = \sigma_v\boldsymbol{a} + \sigma_u\boldsymbol{b} + M\boldsymbol{c} = \boldsymbol{b}_u,$$

the vector-valued 1-form $\boldsymbol{\omega} := \boldsymbol{a}\,du + \boldsymbol{b}\,dv$ is closed. Then by Poincaré's lemma (Theorem 2.6), there exists a vector-valued function $p\colon U \to \mathbb{R}^3$ such that $dp = \boldsymbol{\omega}$:

$$p_u = \boldsymbol{a}, \qquad p_v = \boldsymbol{b}.$$

Let

$$\hat{\mathcal{F}} := (e^{-\sigma}\boldsymbol{a}, e^{-\sigma}\boldsymbol{b}, \boldsymbol{c}).$$

Then it holds that

$$\hat{\mathcal{F}}_u = \hat{\mathcal{F}}\hat{\Omega}, \qquad \hat{\mathcal{F}}_v = \hat{\mathcal{F}}\hat{\Lambda},$$

$$\hat{\Omega} := \begin{pmatrix} 0 & \sigma_v & -e^{-\sigma}L \\ -\sigma_v & 0 & -e^{-\sigma}M \\ e^{-\sigma}L & e^{-\sigma}M & 0 \end{pmatrix}, \qquad \hat{\Lambda} := \begin{pmatrix} 0 & -\sigma_u & -e^{-\sigma}M \\ \sigma_u & 0 & -e^{-\sigma}N \\ e^{-\sigma}M & e^{-\sigma}N & 0 \end{pmatrix}$$

with $\hat{\mathcal{F}}(u_0,v_0) = \mathrm{id}$. Then by Theorem 2.3, $\hat{\mathcal{F}} \in \mathrm{SO}(3)$ for all $(u,v) \in U$. This means that

$$p_u\cdot p_u = \boldsymbol{a}\cdot\boldsymbol{a} = e^{2\sigma}, \qquad p_u\cdot p_v = \boldsymbol{a}\cdot\boldsymbol{b} = 0, \qquad p_v\cdot p_v = \boldsymbol{b}\cdot\boldsymbol{b} = e^{2\sigma}$$
$$p_u\cdot\nu = p_v\cdot\nu = 0, \qquad \nu\cdot\nu = 1,$$

where $\nu := \boldsymbol{c}$. Hence the first fundamental form of p is $ds^2 = e^{2\sigma}(du^2+dv^2)$ and ν is the unit normal vector field of p. Moreover, since

$$p_{uu}\cdot\nu = a_u\cdot\boldsymbol{c} = L, \qquad p_{uv}\cdot\nu = M, p_{vv}\cdot\nu = N.$$

Thus, p is the desired immersion.

Next, we prove the uniqueness. Let $\tilde{p}$ be an immersion with (3.12). Then the Gauss frame $\widetilde{\mathcal{F}}$ satisfies the equation (3.13) as well as $\mathcal{F}$. Here, $|\tilde{p}_u(u_0,v_0)| = e^{\sigma(u_0,v_0)}$, $|\tilde{p}_v(u_0,v_0)| = e^{\sigma(u_0,v_0)}$, and $\tilde{p}_u$, $\tilde{p}_v$, $\tilde{\nu}$ are mutually perpendicular. Thus, by a suitable rotation in $\mathbb{R}^3$, we may assume $\widetilde{\mathcal{F}}(u_0,v_0)$ coincides with $\mathcal{F}(u_0,v_0)$ without loss of generality. Then $\widetilde{F} = \mathcal{F}$ by the uniqueness part of Theorem 2.3, and $dp = d\widetilde{p}$ holds. Hence $\widetilde{p} = p$ up to additive constant vector. $\square$

Exercises

3-1$^{\mathrm{H}}$ Prove Theorem 3.14.

3-2$^{\mathrm{H}}$ Let $(x(u), z(u))$ be a curve on the xz-plane parametrized by the arc-length parameter (that is, $(\dot{x})^2 + (\dot{z})^2 = 1$). Find an isothermal parameter of the surface of revolution

$$p(u, v) = \bigl(x(u)\cos v, x(u)\sin v, z(u)\bigr).$$

The Hopf Differential

Complexification of vector spaces. Let V be an n-dimensional real vector space. By extending the coefficients to complex numbers, we obtain an n-dimensional complex vector space $V^{\mathbb{C}}$, called the *complexification of* V. More precisely, take a basis $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$ of V. Then $V^{\mathbb{C}}$ is the complex vector space generated by $\{\boldsymbol{a}_j\}$:

$$\begin{aligned} V^{\mathbb{C}} &= \{x_1\boldsymbol{a}_1 + \dots + x_n\boldsymbol{a}_n \,|\, x_j \in \mathbb{C} \quad (j = 1, \dots, n)\} \\ &= \operatorname{Span}_{\mathbb{C}}\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}. \end{aligned} \tag{4.1}$$

This expression does not depend on the choice of $\{\boldsymbol{a}_j\}$. In fact, let $\{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$ be another basis of V and $A \in \mathrm{GL}(n, \mathbb{R})$ the change of bases $\{\boldsymbol{a}_j\}$ and $\{\boldsymbol{b}_j\}$:

$$(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)A.$$

Since

$$\begin{aligned} \boldsymbol{x} := x_1\boldsymbol{a}_1 + \dots + x_n\boldsymbol{a}_n &= (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ &= (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad \left(\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} := A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right), \end{aligned}$$

we have that $\operatorname{Span}_{\mathbb{C}}\{\boldsymbol{b}_j\} = \operatorname{Span}_{\mathbb{C}}\{\boldsymbol{a}_j\}$.

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The *dual vector space* W^* of a real (complex) vector space W is the set of linear functions on W:

$$W^* := \{\sigma\colon W \to \mathbb{R} \,|\, \mathbb{R}\text{-linear}\} \quad \bigl(\text{resp.}\{\sigma\colon W \to \mathbb{C} \,|\, \mathbb{C}\text{-linear}\}\bigr).$$

It is easy to see that $(W^{\mathbb{C}})^* = (W^*)^{\mathbb{C}}$.

The complexification $V^{\mathbb{C}}$ is also interpreted as a $2n$-dimensional real vector space spanned by

$$\boldsymbol{a}_1, \dots, \boldsymbol{a}_n; \quad \mathrm{i}\boldsymbol{a}_1, \dots, \mathrm{i}\boldsymbol{a}_n,$$

where $\mathrm{i} = \sqrt{-1}$. Under such a situation, V is an n-dimensional subspace of $V^{\mathbb{C}}$ as a real vector space.

Example 4.1. The complexification of $\mathbb{R}^n$ is $\mathbb{C}^n$. In fact, $\mathbb{C}^n = \operatorname{Span}_{\mathbb{C}}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$, where $\{\boldsymbol{e}_j\}$ is the canonical basis of $\mathbb{R}^n$.

2-dimensional case. We assume that V is a real vector space of dimension 2, and take a basis $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$. Then the *dual basis* $\{\alpha_1, \alpha_2\}$ of V^* is defined by

$$\alpha_j(\boldsymbol{a}_k) = \delta_{jk} = \begin{cases} 1 & (j = k), \\ 0 & (j \neq k) \end{cases},$$

and

$$(V^*)^{\mathbb{C}} = \operatorname{Span}_{\mathbb{C}}(\alpha_1, \alpha_2) = \operatorname{Span}_{\mathbb{C}}(\beta, \bar{\beta}),$$

where

$$\beta := \alpha_1 + \mathrm{i}\alpha_2, \qquad \bar{\beta} := \alpha_1 - \mathrm{i}\alpha_2.$$

We set

$$\boldsymbol{b} := \frac{1}{2}(\boldsymbol{a}_1 - \mathrm{i}\boldsymbol{a}_2), \qquad \bar{\boldsymbol{b}} := \frac{1}{2}(\boldsymbol{a}_1 + \mathrm{i}\boldsymbol{a}_2).$$

Then $\{\boldsymbol{b}, \bar{\boldsymbol{b}}\}$ is a basis of $V^{\mathbb{C}}$ whose dual basis is $\{\beta, \bar{\beta}\}$.

Then a real vector $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 \in V$ is identified with

$$\xi\boldsymbol{b} + \overline{\xi\boldsymbol{b}} = 2\operatorname{Re}(\xi\boldsymbol{b}),$$

where $\xi := x_1 + \mathrm{i}x_2$ and $\bar{\xi}$ is its complex conjugate.

Compexified tangent spaces of Riemann surfaces. Let S be a *Riemann surface*, that is, a complex 1-manifold, and take a local complex coordinate neighborhood $(U; z)$ around $p \in S$. Then (u, v) $(z = u + \mathrm{i}v)$ is a real coordinate system on $U \subset S$.

The tangent space T_xS is a real vector space spanned by $\{(\partial/\partial u)_x, (\partial/\partial v)_x\}$, and $\{(du)_x, (dv)_x\}$ is the dual basis of it. Then, as seen in the previous paragraph, the complexification of $(T_xS)^{\mathbb{C}}$ and its dual $(T_x^*S)^{\mathbb{C}}$is obtained as

$$(4.2) \qquad (T_xS)^{\mathbb{C}} = \operatorname{Span}_{\mathbb{C}}\left\{\left(\frac{\partial}{\partial z}\right)_x, \left(\frac{\partial}{\partial \bar{z}}\right)_x\right\}$$

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial u} - \mathrm{i}\frac{\partial}{\partial v}\right), \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + \mathrm{i}\frac{\partial}{\partial v}\right),$$

$$(4.3) \qquad (T_x^*S)^{\mathbb{C}} = \operatorname{Span}_{\mathbb{C}}\{(dz)_x, (d\bar{z})_x\}$$

$$dz := du + \mathrm{i}dv, \quad d\bar{z} := du - \mathrm{i}dv.$$

In particular $\{(dz)_x, (d\bar{z})_x\}$ is the dual basis of $\{(\partial/\partial z)_x, (\partial/\partial \bar{z})_x\}$.

Lemma 4.2. *Let $(U; z = u + \mathrm{i}v)$ be a complex coordinate neighborhood of a Riemann surface S. Then a function $f\colon U \to \mathbb{C}$ is holomorphic if and only if*

$$\frac{\partial f}{\partial \bar{z}}\left(= \frac{1}{2}\left(\frac{\partial f}{\partial u} - \mathrm{i}\frac{\partial f}{\partial v}\right)\right) = 0.$$

Proof. We write $f(u, v) = \xi(u, v) + \mathrm{i}\eta(u, v)$, where ξ and η are real-valued function on U. Then

$$\begin{aligned} 2\frac{\partial f}{\partial \bar{z}} &= \frac{\partial(\xi + \mathrm{i}\eta)}{\partial u} - \mathrm{i}\frac{\partial(\xi + \mathrm{i}\eta)}{\partial v} \\ &= \left(\frac{\partial \xi}{\partial u} - \frac{\partial \eta}{\partial v}\right) + \mathrm{i}\left(\frac{\partial \eta}{\partial u} + \frac{\partial \xi}{\partial v}\right), \end{aligned}$$

which vanishes if and only if the map $(u, v) \mapsto (\xi, \eta)$ satisfies the Cauchy-Riemann equation. □

Definition 4.3.

$$\begin{aligned} (T_xS)^{(1,0)} &:= \operatorname{Span}_{\mathbb{C}}\{(dz)_x\} \subset (T_x^*S)^{\mathbb{C}}, \\ (T_xS)^{(0,1)} &:= \operatorname{Span}_{\mathbb{C}}\{(d\bar{z})_x\} \subset (T_x^*S)^{\mathbb{C}}. \end{aligned}$$

Lemma 4.4. $(T_x^*S)^{\mathbb{C}} = (T_x^*S)^{(1,0)} \oplus (T_x^*S)^{(0,1)}$. *Moreover such a decomposition does not depend on a choice of complex coordinate systems.*

Proof. Since $(dz)_x$ and $(d\bar{z})_x$ span $(T_x^*(S))^{\mathbb{C}}$, the first part is obtained. Let w be another complex coordinate. Then one can easily show that

$$dw = \frac{\partial w}{\partial z}dz + \frac{\partial w}{\partial \bar{z}}d\bar{z}, \quad d\bar{w} = \frac{\partial \bar{w}}{\partial z}dz + \frac{\partial \bar{w}}{\partial \bar{z}}d\bar{z}.$$

Since the coordinate change $z \mapsto w$ is holomorphic, Lemma 4.2 yields that

$$\frac{\partial w}{\partial \bar{z}} = 0, \qquad \frac{\partial \bar{w}}{\partial z} = \overline{\frac{\partial w}{\partial \bar{z}}} = 0.$$

Hence, by definition of complex derivation,

$$dw = \frac{dw}{dz}\,dz, \qquad d\bar{w} = \overline{\frac{dw}{dz}}\,d\bar{z}$$

hold. Then the second part of the conclusion follows. □

Symmetric 2-differentials on Riemann surfaces. A *symmetric 2-form* on a real vector space V is a bilinear form

$$\sigma\colon V \times V \longrightarrow \mathbb{R}$$

such that $\sigma(\boldsymbol{x}, \boldsymbol{y}) = \sigma(\boldsymbol{y}, \boldsymbol{x})$ holds for all $\boldsymbol{x}$, $\boldsymbol{y} \in V$. A *symmetric 2-tensor* or a *symmetric 2-differential* on a smooth manifold S is a correspondence

$$\sigma\colon S \ni x \longmapsto \text{a symmetric 2-form } \sigma_x \text{ on } T_xS$$

such that $\sigma(X, Y)\colon S \to \mathbb{R}$ is smooth for each smooth vector fields X and Y on S. Taking a local coordinate system (u, v) around p, a symmetric 2-tensor σ is expressed as

$$\begin{aligned}\sigma &= s_{11}\,du^2 + 2s_{12}\,du\,dv + s_{22}\,dv^2 \\ &\left(\begin{matrix} s_{11} := \sigma\left(\partial/\partial u, \partial/\partial u\right), & s_{22} := \sigma\left(\partial/\partial v, \partial/\partial v\right), \\ s_{12} = s_{21} := \sigma\left(\partial/\partial u, \partial/\partial v\right) & \end{matrix}\right).\end{aligned} \tag{4.4}$$

Example 4.5 (Surfaces in the Euclidean space)**.** Let $p\colon S \to \mathbb{R}^3$ be an immersion of a Riemann surface S into $\mathbb{R}^3$. Since S is

orientable,[9] there exists a (globally defined) unit normal vector field ν which is considered as a map $\nu\colon S \to S^2 \subset \mathbb{R}^3$, called the *Gauss map*.

The *first fundamental form* ds^2 and the *second fundamental form* $I\!I$ are defined as

$$ds^2(\boldsymbol{v}, \boldsymbol{w}) := dp(\boldsymbol{v}) \cdot dp(\boldsymbol{w}) \text{ and } I\!I(\boldsymbol{v}, \boldsymbol{w}) := -dp(\boldsymbol{v}) \cdot d\nu(\boldsymbol{w}),$$

respectively, for $\boldsymbol{v}$, $\boldsymbol{w} \in T_xS$ $(x \in S)$. Then both ds^2 and $I\!I$ are symmetric 2-differentials on S.

Since $dp(\partial/\partial u) = p_u, \dots,$ and

$$\begin{aligned} &p_u \cdot \nu_u = (p_u \cdot \nu)_u - p_{uu} \cdot \nu, \\ &p_u \cdot \nu_v = p_v \cdot \nu_u = -p_{uv} \cdot \nu, \quad p_v \cdot \nu_v = -p_{vv} \cdot \nu, \end{aligned}$$

the definitions of the fundamental forms here coincide with those as (2.11) in Section 2.

Let $(U; z = u + \mathrm{i}v)$ be a complex chart of a Riemann surface S. By virtue of (4.3), one can rewrite (4.4) as

$$\sigma = \tilde{s}_{20}\,dz^2 + 2\tilde{s}_{11}\,dz\,d\bar{z} + \tilde{s}_{02}d\bar{z}^2, \tag{4.5}$$

where[10]

$$\begin{aligned} &\tilde{s}_{20} = \frac{s_{11} - s_{22} - 2\mathrm{i}s_{12}}{4}, \\ &\tilde{s}_{02} = \frac{s_{11} - s_{22} + 2\mathrm{i}s_{12}}{4}, \qquad \tilde{s}_{11} = \frac{s_{11} + s_{22}}{4}. \end{aligned}$$

[9] A Riemann surface (more generally, a complex manifold) is necessarily orientable. In fact, a holomorphic coordinate change $z = u + \mathrm{i}v \mapsto w = \xi + \mathrm{i}\eta$ has positive Jacobian because of the Cauchy-Riemann equation.

[10] Although the form (4.5) might be written as $\sigma^{\mathbb{C}}$ because it is a complexification of the original σ, we do not distinguish them in this notebook.

Definition 4.6. Let σ be a symmetric 2-differential as in (4.5). Then we set

$$\sigma^{(2,0)} := \tilde{\sigma}_{20} dz^2,\ \sigma^{(1,1)} := 2\tilde{\sigma}_{11} dz\, d\bar{z},\ \sigma^{(0,2)} := 2\tilde{\sigma}_{02} d\bar{z}^2,$$

and call them the $(2,0)$-part, $(1,1)$-part, and $(0,2)$-part of σ, respectively.

Similar to Lemma 4.4,

Lemma 4.7. *The $(2,0)$-part, $(1,1)$-part and $(0,2)$-part of symmetric 2-differnetials are independent on choice of complex coordinates.*

Hopf differentials.

Definition 4.8. An immersion $p\colon S \to \mathbb{R}^3$ is said to be *conformal* if each complex coordinate $z = u + \mathrm{i}v$ corresponds to isothermal coordinate system (u, v).

In the situation of Definition 4.8, the first fundamental form ds^2 is written as

$$ds^2 = e^{2\sigma}(du^2 + dv^2) = e^{2\sigma}\, dz\, d\bar{z}. \tag{4.6}$$

Thus we have

Lemma 4.9. *An immersion $p\colon S \to \mathbb{R}^3$ of a Riemann surface S is conformal if and only if the first fundamental form has no both $(2,0)$-part and $(0,2)$-part.*

Definition 4.10. Let $p\colon S \to \mathbb{R}^3$ be a conformal immersion of a Riemann surface of S. The $(2,0)$-part Q of the second fundamental form is called the *Hopf differential.*

Lemma 4.11. *If the first and second fundamental forms are in the form*

$$\begin{aligned} ds^2 &= e^{2\sigma}(du^2 + dv^2) = e^{2\sigma}\, dz\, d\bar{z},\\ I\!I &= L\, du^2 + 2M\, du\, dv + N\, dv^2 \end{aligned} \tag{4.7}$$

in the complex coordinate $z = u + \mathrm{i}v$, the Hopf differential Q and the mean curvature H are expressed as

$$Q = \frac{1}{4}\big((L - N) - 2\mathrm{i}M\big)\, dz^2, \qquad H = \frac{e^{-2\sigma}}{2}(L + N). \tag{4.8}$$

Proof. The equation **??** yields the expression of the Hopf differential. Since the representation matrix of the first fundamental form is $e^{2\sigma}\,\mathrm{id}$, then the coefficients of the Weingarten matrix (cf. (**??**) in Section 2) are $e^{-2\sigma}$ times of L, M and N. Since the $2H$ is the trace of the Weingarten matrix, the expression of the mean curvature holds. □

Definition 4.12. Let $p\colon S \to \mathbb{R}^3$ be an immersion of a 2-manifold S. A point $x \in S$ is called an *umbilic point* if the first fundamental form ds^2 and the second fundamental form $I\!I$ are proportional at the point p. If all points of S are umbilic points, p is called *totally umbilic.*

Proposition 4.13 (cf. §7 in [3-1])**.** *The image of a totally umbilic immersion is a part of a plane or a round sphere.*

Proof. Since the first and second fundamental forms are proportional, the Weingarten matrix (**??**) is a scalar multiplication of id: $A = \lambda\,\mathrm{id}$ on a coordinate neighborhood (u, v). Then the derivatives of the unit normal vector field satisfy

$$\nu_u = -\lambda p_u, \qquad \nu_v = -\lambda p_v.$$

Differentiating these, we have

$$\begin{aligned}\nu_{uv} &= -\lambda_v p_u + \lambda p_{uv},\\ \nu_{vu} &= -\lambda_u p_v + \lambda p_{vu}.\end{aligned}$$

This implies $d\lambda = 0$ on a coordinate neighborhood, and thus λ must be constant. When $\lambda = 0$, ν is constant vector, and then the image of p is a part of the plane. If $\lambda \neq 0$, $p + \nu/\lambda$ is constant. This means that the image lies on a sphere of radius $1/|\lambda|$. □

The Gauss and Codazzi equations.

Theorem 4.14. *Let $p\colon S \to \mathbb{R}^3$ be a conformal immersion of a Riemann surface S, and let ds^2, H and Q be the first fundamental form, the mean curvature and the Hopf differential, respectively. Take a complex coordinate $z = u + \mathrm{i}v$ of S, and write*

$$ds^2 = e^{2\sigma}\,dz\,d\bar z, \quad Q = q\,dz^2.$$

Then the Gauss equation (3.14) *and the Codazzi equations* (3.15) *are equivalent to*

$$\frac{\partial^2\sigma}{\partial z\partial\bar z} + e^{-2\sigma} q\bar q + \frac{1}{4}e^{2\sigma}H^2 = 0, \qquad \frac{\partial q}{\partial \bar z} = \frac{e^{2\sigma}}{4}\frac{\partial H}{\partial z}, \tag{4.9}$$

respectively.

Proof. By (4.8),

$$\begin{aligned} q\bar q &= \frac{1}{16}\bigl((L-N)^2 + 4M^2\bigr) = \frac{1}{16}\left((L+N)^2 - 4(LN - M^2)\right)\\ &= \frac{1}{4}\left(e^{4\sigma}H^2 - (LN - M^2)\right).\end{aligned}$$

Since

$$\frac{\partial^2}{\partial z\partial\bar z} = \frac{1}{4}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right),$$

the Gauss equation (3.14) is equivalent to the first equation of (4.9). The second equation follows from (3.15). □

Corollary 4.15. *Let $p\colon S \to \mathbb{R}^3$ be a conformal immersion of a Riemann surface S with constant mean curvature. Then the Hopf differential $Q = q\,dz^2$ is holomorphic, that is, q is a holomorphic function in z, where z is an arbitrary complex coordinate on S.*

Proof. When $dH = 0$, the second equation of (4.9) implies $q_{\bar z} = 0$. □

Since zeros of holomorhpic function are isolated unless the function is identically zero, we have

Corollary 4.16. *An umbilic point of a constant mean curvature surface is isolated unless it is totally umbilic.*

References

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Exercises

4-1$^{\mathrm{H}}$ Let S be a Riemann surface, and let

$$p\colon S \longrightarrow \mathbb{R}^3$$

be a conformal immersion of constant mean curvature without umbilic points. Then for each $x \in D$, there exists a complex coordinate z such that

$$ds^2 = e^{2\sigma}\, dz\, d\bar{z}, \qquad Q = dz^2.$$