Examples of Constant Mean Curvature Surfaces

Planar curves. Let $\gamma\colon I \ni s \mapsto \gamma(s) \in \mathbb{R}^2$ be a smooth map defined on an $I \subset \mathbb{R}$. Then γ is called a *regular curve* if $\dot\gamma \neq 0$ on I, where $\dot{} = d/ds$. The parameter s is called an *arc length parameter* if

$$|\dot\gamma(s)| = \left|\frac{d\gamma}{ds}(s)\right| = 1 \tag{3.1}$$

holds on I.

Lemma 3.1. *A regular curve $\gamma\colon I \ni t \mapsto \gamma(t) \in \mathbb{R}^2$ defined on an interval $I \subset \mathbb{R}$ can be reparametrized by an arc length parameter. Moreover, such an arc length parameter is unique up to additive constants.*

Proof. Fix $t_0 \in I$ and define a function $s\colon I \to \mathbb{R}$ by

$$s(t) := \int_{t_0}^{t} \left|\frac{d\gamma}{dt}(u)\right| du.$$

Then $s\colon I \to J \subset \mathbb{R}$ is a smooth function such that $ds/dt > 0$. Hence there exists the smooth inverse $J \ni s \mapsto t(s) \in I$. Then $\tilde\gamma(s) := \gamma\bigl(t(s)\bigr)$ is the desired reparametrization. In fact,

$$\begin{aligned}\left|\frac{d\tilde\gamma(s)}{ds}\right| &= \left|\frac{d\gamma}{dt}\bigl(t(s)\bigr)\frac{dt}{ds}(s)\right| = \left|\frac{d\gamma}{dt}\bigl(t(s)\bigr)\frac{1}{ds/dt(t(s))}\right| \\ &= \left|\frac{d\gamma}{dt}\bigl(t(s)\bigr)\frac{1}{|d\gamma/dt\bigl(t(s)\bigr)|}\right| = 1.\end{aligned}$$

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So we have the first assertion. Let s and u be two arc length parameters. Then there exists a parameter change $u = u(s)$, which is strictly increasing function such that

$$1 = \left|\frac{d\gamma}{ds}\right| == \left|\frac{d\gamma}{du}\frac{du}{ds}\right| = \frac{du}{ds}\left|\frac{d\gamma}{du}\right| = \frac{du}{ds}.$$

Hence $u = s + \text{constant}$, proves the second assertion. □

Throughout this section, we assume that planar curves are parameterized by arc length parameter.

Let $\gamma(s) = {}^t\bigl(x(s), y(s)\bigr)$ $(s \in I)$be a parametrized planar curve where s is an arc length parameter. Then

$$\boldsymbol{e}(s) := \dot\gamma(s) = \begin{pmatrix}\dot x(s)\\ \dot y(s)\end{pmatrix}, \qquad \boldsymbol{n}(s) := \begin{pmatrix}-\dot y(s)\\ \dot x(s)\end{pmatrix}$$

are mutually perpendicular orthogonal vectors for each $s \in I$. Thus we have obtained a map

$$\mathcal{F}(s) := \bigl(\boldsymbol{e}(s), \boldsymbol{n}(s)\bigr)\colon I \longmapsto \mathrm{SO}(2), \tag{3.2}$$

where SO(2) is the set (a group) of 2×2-orthogonal matrix of determinant 1. We call $\mathcal{F}$ the *frame* of γ. Note that

$$\mathrm{SO}(2) = \{R(\theta) \,|\, \theta \in \mathbb{R}\}, \quad R(\theta) := \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}. \tag{3.3}$$

Theorem 3.2 (The Frenet formula)**.** *Let $\mathcal{F}(s)$ be the frame of the curve $\gamma(s)$ where s is an arc length parameter defined on an* *interval I. Then there exists a unique smooth function $\kappa\colon I \to \mathbb{R}$ such that*

$$\dot{\mathcal{F}} = \mathcal{F}\Omega \qquad \Omega(s) := \kappa(s)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.4}$$

Proof. Since $\mathcal{F}$ is a function valued on $\mathrm{SO}(2)$, $\mathcal{F}^{-1}\dot{\mathcal{F}}$ is valued on the set of skew-symmetric matrices. In fact, since ${}^t\mathcal{F} = \mathcal{F}^{-1}$,

$$\begin{aligned} {}^t\left(\mathcal{F}^{-1}\dot{F}\right) &= {}^t\left({}^t\mathcal{F}\dot{\mathcal{F}}\right) = {}^t\dot{\mathcal{F}}\mathcal{F} = \frac{d}{ds}\mathcal{F}^{-1}\mathcal{F} \\ &= -\mathcal{F}^{-1}\dot{\mathcal{F}}F^{-1}\mathcal{F} = -\mathcal{F}^{-1}\dot{\mathcal{F}}. \end{aligned}$$

Hence there exists a function $\kappa(s)$ such that

$$\mathcal{F}^{-1}\dot{\mathcal{F}} = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix},$$

proving the theorem. □

We call the function κ the *curvature* of the curve γ.

Proposition 3.3. *Let $\gamma(s) = {}^t\big(x(s), y(s)\big)$ be a planar curve parametrized by the arc length s. Then its curvature satisfies*

$$\kappa = \dot{x}\ddot{y} - \dot{y}\ddot{x}.$$

Theorem 3.4 (The fundamental theorem for planar curves)**.** *Let $\kappa\colon I \to \mathbb{R}$ be a smooth function. Then there exists a curve $\gamma\colon I \to \mathbb{R}$ parametrized by the arc length whose curvature is κ. Moreover, such a curve γ is unique up to rotations and translations of $\mathbb{R}^2$.*

Proof. First we shall prove uniqueness: Let γ_j $(j = 1, 2)$ be curves with curvature κ, and denote by $\mathcal{F}_j$ $(j = 1, 2)$ the frame of γ_j. Then by (3.4),

$$\begin{aligned} \frac{d}{ds}(\mathcal{F}_2\mathcal{F}_1^{-1}) = \frac{d}{ds}(\mathcal{F}_2{}^t\mathcal{F}_1) &== \dot{\mathcal{F}}_2{}^t\mathcal{F}_1 + \mathcal{F}_2{}^t\dot{\mathcal{F}}_1 \\ &= \mathcal{F}_2\Omega^t\mathcal{F}_1 + \mathcal{F}_2{}^t\mathcal{F}_1\Omega = \mathcal{F}_2(\Omega + {}^t\Omega)^t\mathcal{F}_1 = O \end{aligned}$$

holds, and thus there exist constant matrix such that

$$\mathcal{F}_2\mathcal{F}_1^{-1} = A \qquad (A \in \mathrm{SO}(2)),$$

that is, $\mathcal{F}_2 = A\mathcal{F}_1$. Comparing the first column of this, we have

$$\dot{\gamma}_2 = A\dot{\gamma}_1 \qquad \text{and then} \qquad \gamma_2 = A\dot{\gamma}_1 + \boldsymbol{a},$$

where $A \in \mathrm{SO}(2)$ and $\boldsymbol{a} \in \mathbb{R}^2$. Hence the uniqueness part holds.

Next, we prove existence: fix $s_0 \in I$ and set

$$\gamma(s) := \int_{s_0}^{s}\left(\cos\int_{s_0}^{u}\kappa(t)\,dt, \sin\int_{s_0}^{u}\kappa(t)\,dt\right)du.$$

Then one can check that s is the arc length parameter of $\gamma(s)$, and $\kappa(s)$ is the curvature. □

Surfaces of revolution. Let $\gamma(s) = \big(x(s), y(s)\big)$ be a regular curve parametrized by the arc length s, satisfying $y(s) > 0$ for all s. Then the *surface of revolution* of γ about the x-axis is parametrized as

$$f(t, s) := \big(x(s), y(s)\cos t, y(s)\sin t\big), d \qquad (t, s) \in S^1 \times I. \tag{3.5}$$

The curve γ is called the *profile curve* of the surface (3.5).

Noticing $\dot{x}^2+\dot{y}^2=1$, the first fundamental form I and the second fundamental form of f are expressed as

$$I = y^2\,dt^2+ds^2, \quad II = -\dot{x}y\,dt^2+(\dot{x}\ddot{y}-\dot{y}\ddot{x})\,ds^2 = -\dot{x}y\,dt^2+\kappa\,ds^2,$$

where κ is the curvature of the profile curve (cf. Proposition 3.3). Hence we have

Proposition 3.5. *The mean curvature function H of the surface* (3.5) *is expressed as*

$$2H = \kappa - \frac{\dot{x}}{y}. \tag{3.6}$$

Delaunay surfaces.

Theorem 3.6. *Let H be a non-zero constant. Then the profile curve $(x(s), y(s))$ of a surface of revolution with constant mean curvature H is expressed as*

$$\begin{aligned} y(s) &= \frac{1}{2H}\sqrt{(2Ha+1)^2-2(2Ha+1)\cos 2Hs}, \\ x(s) &= \int_0^s \frac{1+(2aH+1)\cos 2Hu}{y(u)}\,du, \end{aligned} \tag{3.7}$$

up to horizontal translations and parameter changes, where a is a constant.

Proof. Let $\gamma(s) := \bigl(x(s), y(s)\bigr)$ be the profile curve of given surface of revolution with constant mean curvature H. Then by

(3.6), the curvature function κ of γ satisfies

$$\kappa = 2H + \frac{\dot{x}}{y}.$$

Thus, the frame $\mathcal{F}$ of γ satisfies the Frenet formula (Theorem 3.2):

$$\dot{\mathcal{F}} = \left(2H+\frac{\dot{x}}{y}\right)\mathcal{F}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{3.8}$$

We shall find the curve solving this differential equation. Set

$$\widetilde{\mathcal{F}} := y\mathcal{F}.$$

Then, noticing

$$\dot{x}^2+\dot{y}^2=1, \tag{3.9}$$

the equation (3.8) is equivalent to

$$\dot{\widetilde{\mathcal{F}}} = 2H\mathcal{F}\begin{pmatrix}0 & -2H\\ 2H & 0\end{pmatrix} + \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{3.10}$$

Let

$$\begin{aligned} &A(s) := \widetilde{\mathcal{F}}(s)\mathcal{F}_0(s)^{-1}, \\ &\mathcal{F}_0(s) := R(2Hs) = \begin{pmatrix}\cos 2Hs & -\sin 2Hs\\ \sin 2Hs & \cos 2Hs\end{pmatrix}. \end{aligned} \tag{3.11}$$

Substituting $\widetilde{\mathcal{F}} = A\mathcal{F}_0$ into (3.10), we have

$$\dot{A} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathcal{F}_0^{-1} = \begin{pmatrix}\sin 2Hs & -\cos 2Hs\\ \cos 2Hs & \sin 2Hs\end{pmatrix}, \tag{3.12}$$

and then

$$A = \frac{-1}{2H}\begin{pmatrix} \cos 2Hs & \sin 2Hs \\ -\sin 2Hs & \cos 2Hs \end{pmatrix} + C, \tag{3.13}$$

where C is a constant matrix. Summing up, it holds that

$$\begin{aligned} y\mathcal{F} = \widetilde{\mathcal{F}} &= A\mathcal{F}_0 \\ &= \frac{1}{2H}\left(C\begin{pmatrix} \cos 2Hs & -\sin 2Hs \\ \sin 2Hs & \cos 2Hs \end{pmatrix} - \mathrm{id}\right). \end{aligned} \tag{3.14}$$

Since right-hand side is a periodic function and $\mathcal{F} \in \mathrm{SO}(2)$, y^2 (and then y) is a periodic function. Hence y must take both maximum and minimum. By a change of parameter s to $s+$ constant and a horizontal translation $x \mapsto x+$constant, we may assume y takes its maximum at $s = 0$, and $x(0) = 0$. Moreover, by the reflection of the y-axis, we may assume $\dot{x}(0) \geqq 0$ without loss of generality. Hence we can assume an initial condition

$$\bigl(x(0), y(0)\bigr) = (0, a), \quad \bigl(\dot{x}(0), \dot{y}(0)\bigr) = (1, 0), \quad \ddot{y}(0) = \kappa(0) \leqq 0.$$

Substituting these into (3.14), we have $C = (2Ha + 1)\,\mathrm{id}$:

$$y\mathcal{F} = \frac{1}{2H}\left((2Ha+1)\begin{pmatrix} \cos 2Hs & -\sin 2Hs \\ \sin 2Hs & \cos 2Hs \end{pmatrix} - \mathrm{id}\right). \tag{3.15}$$

Taking the determinant of this, we have

$$y^2 = \frac{1}{(2H)^2}\bigl((2Ha+1)\cos 2Hs - 1)^2 + (2Ha^2+1)\sin^2 2Hs\bigr)$$

and then

$$y = \frac{1}{2H}\sqrt{(2Ha+1)^2 - 2(2Ha+1)\cos 2Hs}.$$

On the other hand, the $(1,1)$-component of (**??**) is expressed as

$$y\dot{x} = \frac{1}{2H}\bigl(1 + (2aH+1)\cos 2Hs\bigr).$$

Thus we have the conclusion. □

The surfaces in (3.7) are called the *Delaunay surfaces*.

References

[3-1] 劔持勝衛：「曲面論講義 — 平均曲率一定曲面入門」（培風館，2000）.

[3-2] K. Kenmotsu, SURFACES WITH CONSTANT MEAN CURVATURE, Translations of Mathematical Monographs, translated by Katsuhiro Moriya, American Math. Soc., 2003.

Exercises

3-1$^{\mathrm{H}}$ Draw pictures of Delaunay curves for $H = \frac{1}{2}$.

3-2$^{\mathrm{H}}$ Classify minimal surfaces of revolution.