

1 Area minimizing surfaces

1.1 A review of surface theory.

Let $D \subset \mathbb{R}^2$ be a domain in the uv -plane and $f: D \rightarrow \mathbb{R}^3$ an immersion. We often refer to such an immersion as a *surface*. Then the *unit normal vector* of f is given by (with \pm -ambiguity)

$$(1.1) \quad \nu := \frac{f_u \times f_v}{|f_u \times f_v|} : D \longrightarrow S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\} \subset \mathbb{R}^3,$$

where “ \times ” denotes the vector product of \mathbb{R}^3 . The *first* and the *second fundamental forms* are defined as

$$(1.2) \quad \begin{aligned} ds^2 &= df \cdot df = E du^2 + 2F du dv + G dv^2, \\ II &= -df \cdot d\nu = L du^2 + 2M du dv + N dv^2, \end{aligned}$$

where “ \cdot ” denotes the canonical inner product of \mathbb{R}^3 . Here,

$$\begin{aligned} E &:= f_u \cdot f_u, & F &:= f_u \cdot f_v = f_v \cdot f_u, & G &:= f_v \cdot f_v, \\ L &:= -f_u \cdot \nu_u, & M &:= -f_u \cdot \nu_v = -f_v \cdot \nu_u, & N &:= -f_v \cdot \nu_v \\ &= f_{uu} \cdot \nu, & &= f_{uv} \cdot \nu, & &= f_{vv} \cdot \nu \end{aligned}$$

are called the *entries of the first and the second fundamental forms* with respect to the parameters (u, v) . The *area* of the image of a compact region $\Omega \subset D$ is computed as

$$(1.3) \quad \mathcal{A}(\Omega) := \iint_{\Omega} dA = \iint_{\Omega} |f_u \times f_v| du dv,$$

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where $dA = |f_u \times f_v| du dv = \sqrt{EG - F^2} du dv$ is said to be the *area element* of the surface.

The derivatives of ν is written as (the Weingarten Formula)

$$(1.4) \quad \nu_u = -A_1^1 f_u - A_1^2 f_v, \quad \nu_v = -A_2^1 f_u - A_2^2 f_v,$$

$$A := \begin{pmatrix} A_1^1 & A_1^2 \\ A_2^1 & A_2^2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

The matrix A is called the *Weingarten matrix*, and the determinant K and the half H of the trace of A are called the *Gaussian curvature* and the *mean curvature*, respectively:

$$(1.5) \quad K := \det A = \frac{LN - M^2}{EG - F^2}, \quad H := \frac{1}{2} \operatorname{tr} A = \frac{A_1^1 + A_2^2}{2}.$$

1.2 Area minimizing surfaces.

The purpose of this section is to show the following fact:

For a given simple closed curve C in \mathbb{R}^3 , the surface which minimizing area among all surfaces bounded by C is a surface whose mean curvature vanishes identically.

Setting up. As the description of the above fact is rather intuitive, we will formulate the problem.

Let C be a simple closed smooth curve in \mathbb{R}^3 and set

$$(1.6) \quad \mathcal{S}_C := \left\{ f: \overline{D} \rightarrow \mathbb{R}^3; \begin{array}{l} f \text{ is a } C^\infty\text{-immersion} \\ f(\partial D) = C \end{array} \right\},$$

where D (resp. \overline{D}) is the open (resp. closed) unit disc and ∂D is its boundary:¹

$$(1.7) \quad \overline{D} := D \cup \partial D, \quad D := \{(u, v) \in \mathbb{R}^2; u^2 + v^2 < 1\}, \\ \partial D := \{(u, v) \in \mathbb{R}^2; u^2 + v^2 = 1\} \\ = \{(\cos \theta, \sin \theta); \theta \in \mathbb{R}\}.$$

Roughly speaking, \mathcal{S}_C is “the set of the surfaces bounded by C ”. Then we set the *area functional* as

$$(1.8) \quad \mathcal{A}: \mathcal{S}_C \ni f \mapsto \mathcal{A}(f) = \iint_{\overline{D}} |f_u \times f_v| \, du \, dv.$$

Using these notations, our result can be stated as the following:

Theorem 1.1. *If a surface $f \in \mathcal{S}_C$ attains the minimum of the area functional \mathcal{A} , the mean curvature of f vanishes identically.*

Taking this fact into account, we define

Definition 1.2. A surface whose mean curvature vanishes identically is said to be *minimal*.

Remark 1.3. As Theorem 1.1 is a necessary condition for the minimizer, a minimal surface is not necessarily a minimizer of the area functional.

¹A map f defined on \overline{D} is said to be C^∞ if there exists a open set \tilde{D} containing \overline{D} and a C^∞ map \tilde{f} defined on \tilde{D} such that $\tilde{f}|_{\overline{D}} = f$.

Variations of surfaces. To show Theorem 1.1, we want to “differentiate” the functional \mathcal{A} .

Definition 1.4. For a surface $f \in \mathcal{S}_C$, a *variation* (fixing the boundary) of f is a C^∞ -map

$$\mathcal{F}: \overline{D} \times (-\varepsilon, \varepsilon) \ni (u, v; t) \mapsto f^t(u, v) := \mathcal{F}(u, v; t) \in \mathbb{R}^3$$

such that $f^0 = f$ and $f^t \in \mathcal{S}_C$ for each $t \in (-\varepsilon, \varepsilon)$, where ε is a positive number. The vector-valued function

$$(1.9) \quad V(u, v) := \left. \frac{\partial}{\partial t} \right|_{t=0} f^t(u, v)$$

is called the *variational vector field* of the variation \mathcal{F} .

Lemma 1.5. *For a variation $\mathcal{F} = \{f^t\}$ of $f \in \mathcal{S}_C$ with variational vector field V , it holds that*

$$\frac{d}{d\theta} f(\cos \theta, \sin \theta) \times V(\cos \theta, \sin \theta) = \mathbf{0}.$$

Proof. Since $(\cos \theta, \sin \theta)$ is a parametrization of ∂D , $\gamma^t(\theta) := f^t(\cos \theta, \sin \theta) \in C$ for all t and θ . Thus, two vectors in the left-hand side of the first assertion are both tangent to C , proving the lemma. \square

The first variation formula.

Theorem 1.6. *Let $\mathcal{F} = \{f^t\}$ be a variation of $f \in \mathcal{S}_C$ with variational vector field V . Then it holds that*

$$(1.10) \quad \left. \frac{d}{dt} \right|_{t=0} \mathcal{A}(f^t) = -2 \iint_{\overline{D}} H(V \cdot \nu) \, dA,$$

where H , ν and dA are the mean curvature, the unit normal vector and the area element of f , respectively.

Proof. By the definition of the area (1.3), we have

$$\begin{aligned}
 (*) &:= \frac{d}{dt} \Big|_{t=0} \mathcal{A}(f^t) = \frac{d}{dt} \Big|_{t=0} \iint_{\overline{D}} |f_u^t \times f_v^t| du dv \\
 &= \iint_{\overline{D}} \frac{\partial}{\partial t} \Big|_{t=0} |f_u^t \times f_v^t| du dv \\
 &= \iint_{\overline{D}} \frac{(V_u \times f_v + f_u \times V_v) \cdot (f_u \times f_v)}{|f_u \times f_v|} du dv \\
 &= \iint_{\overline{D}} (V_u \times f_v + f_u \times V_v) \cdot \nu du dv \\
 &= \iint_{\overline{D}} ((V_u \times f_v) \cdot \nu + (f_u \times V_v) \cdot \nu) du dv.
 \end{aligned}$$

Here, by the formula of *scalar triple product*

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = \det(\mathbf{a}, \mathbf{b}, \mathbf{c}),$$

we have

$$\begin{aligned}
 (*) &= \iint_{\overline{D}} ((\nu \times f_v) \cdot V_u + (f_u \times \nu) \cdot V_v) du dv \\
 &= (II) - (I), \\
 (I) &:= \iint_{\overline{D}} [((\nu \times f_v) \cdot V)_u + ((f_u \times \nu) \cdot V)_v] du dv, \\
 (II) &:= \iint_{\overline{D}} [((\nu \times f_v)_u \cdot V) + (f_u \times \nu)_v \cdot V)] du dv.
 \end{aligned}$$

By the Green-Stokes formula, (I) is computed as

$$\begin{aligned}
 (I) &= \iint_{\overline{D}} [((\nu \times f_v) \cdot V)_u - ((\nu \times f_u) \cdot V)_v] du dv, \\
 &= \int_{\partial D} \nu \cdot ((f_u du + f_v dv) \times V) \\
 &= \int_{-\pi}^{\pi} \nu \cdot \left(\frac{d}{d\theta} f(\cos \theta, \sin \theta) \times V(\cos \theta, \sin \theta) \right) d\theta = 0.
 \end{aligned}$$

Here, the last assertion is obtained by Lemma 1.5. On the other hand, using the Weingarten formula (1.4), (II) is computed as

$$\begin{aligned}
 (II) &:= \iint_{\overline{D}} [(\nu_u \times f_v) \cdot V + (\nu \times f_{vu}) \cdot V \\
 &\quad + (f_{uv} \times \nu) \cdot V + (f_u \times \nu_v) \cdot V] du dv \\
 &= \iint_{\overline{D}} [(\nu_u \times f_v) \cdot V + (f_u \times \nu_v) \cdot V] du dv \\
 &= - \iint_{\overline{D}} [((A_1^1 f_u + A_1^2 f_v) \times f_v) \cdot V \\
 &\quad + (f_u \times (A_2^1 f_u + A_2^2 f_v)) \cdot V] du dv \\
 &= - \iint_{\overline{D}} (A_1^1 + A_2^2)(f_u \times f_v) \cdot V du dv \\
 &= - \iint_{\overline{D}} 2H(\nu \cdot V)|f_u \times f_v| du dv \quad \square
 \end{aligned}$$

Proof of Theorem 1.1. We need the following “the fundamental lemma for calculus of variations”.

Lemma 1.7. Assume a smooth function $h: \overline{D} \rightarrow \mathbb{R}$ satisfies

$$\iint_{\overline{D}} h(u, v) \varphi(u, v) du dv = 0$$

for all C^∞ -function with $\varphi|_{\partial D} = 0$. Then $h = 0$ on D .

Proof. Assume $h(u_0, v_0) > 0$ (resp. < 0) $((u_0, v_0) \in D)$. By a continuity, there exists $\varepsilon > 0$ such that $h(u, v) > -$ on an ε -ball $B := B_\varepsilon(u_0, v_0)$ centered at (u_0, v_0) . Let φ be a non-negative C^∞ -function on \overline{D} such that $\varphi > 0$ on B and 0 on $\overline{D} \setminus B$. Then

$$\iint_{\overline{D}} h \varphi du dv = \iint_B h \varphi du dv > 0 \quad (\text{resp. } < 0),$$

a contradiction. \square

Proof of Theorem 1.6. Assume $f \in \mathcal{S}_C$ minimizes the area. Then for any variation $\mathcal{F} = \{f^t\}$ of f , $\mathcal{A}(f^t)$ is not less than $\mathcal{A}(f) = \mathcal{A}(f^0)$. Then by Theorem 1.6, it holds that

$$0 = \left. \frac{d}{dt} \right|_{t=0} \mathcal{A}(f^t) = -2 \int_{\overline{D}} H(V \cdot \nu) |f_u \times f_v| du dv.$$

Let φ be a C^∞ -function on \overline{D} with $\varphi|_{\partial D} = 0$. Then $f^t := f + t\varphi\nu$ is a variation of f with variational vector field $V = \varphi\nu$. Thus,

$$\iint H |f_u \times f_v| \varphi = 0.$$

Since φ is arbitrary, Lemma 1.7 yields the conclusion. \square

Exercises

1-1^H Define a functional $\mathcal{V}: \mathcal{S}_C \rightarrow \mathbb{R}$ defined on \mathcal{S}_C as in (1.6) as

$$\mathcal{V}(f) := \frac{1}{3} \iint_{\overline{D}} (f \cdot \nu) |f_u \times f_v| du dv \quad (f \in \mathcal{S}_C).$$

Then

(1) Explain a geometric meaning of $\mathcal{V}(f)$.

(2) Compute $\left. \frac{d}{dt} \right|_{t=0} \mathcal{V}(f^t)$ for a variation $\{f^t\}$ of $f \in \mathcal{S}_C$.