Area minimizing surfaces

A review of surface theory.

Let $D \subset \mathbb{R}^2$ be a domain in the uv-plane and $f\colon D \to \mathbb{R}^3$ an immersion. We often refer to such an immersion as a *surface*. Then the *unit normal vector* of f is given by (with $\pm$-ambiguity)

$$\nu := \frac{f_u \times f_v}{|f_u \times f_v|} \colon D \longrightarrow S^2 = \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, |\boldsymbol{x}| = 1\} \subset \mathbb{R}^3, \tag{1.1}$$

where "$\times$" denotes the vector product of $\mathbb{R}^3$. The *first* and the *second fundamental forms* are defined as

$$\begin{aligned} ds^2 &= df \cdot df = E\,du^2 + 2F\,du\,dv + G\,dv^2, \\ II &= -df \cdot d\nu = L\,du^2 + 2M\,du\,dv + N\,dv^2, \end{aligned} \tag{1.2}$$

where "$\cdot$" denotes the canonical inner product of $\mathbb{R}^3$. Here,

$$\begin{aligned} E &:= f_u \cdot f_u, & F &:= f_u \cdot f_v = f_v \cdot f_u, & G &:= f_v \cdot f_v, \\ L &:= -f_u \cdot \nu_u, & M &:= -f_u \cdot \nu_v = -f_v \cdot \nu_u, & N &:= -f_v \cdot \nu_v \\ &= f_{uu} \cdot \nu, & &= f_{uv} \cdot \nu, & &= f_{vv} \cdot \nu \end{aligned}$$

are called the *entries of the first and the second fundamental forms* with respect to the parameters (u, v). The *area* of the image of a compact region $\Omega \subset D$ is computed as

$$\mathcal{A}(\Omega) := \iint_\Omega dA = \iint_\Omega |f_u \times f_v|\,du\,dv, \tag{1.3}$$

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where $dA = |f_u \times f_v|\,du\,dv = \sqrt{EG - F^2}\,du\,dv$ is said to be the *area element* of the surface.

The derivatives of ν is written as (the Weingarten Formula)

$$\begin{gathered} \nu_u = -A^1_1 f_u - A^2_1 f_v, \qquad \nu_v = -A^1_2 f_u - A^2_2 f_v, \\ A := \begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}. \end{gathered} \tag{1.4}$$

The matrix A is called the *Weingarten matrix*, and the determinant K and the half H of the trace of A are called the *Gaussian curvature* and the *mean curvature*, respectively:

$$K := \det A = \frac{LN - M^2}{EG - F^2}, \qquad H := \frac{1}{2}\operatorname{tr} A = \frac{A^1_1 + A^2_2}{2}. \tag{1.5}$$

Area minimizing surfaces.

The purpose of this section is to show the following fact:

> For a given simple closed curve C in $\mathbb{R}^3$, the surface which minimizing area among all surfaces bounded by C is a surface whose mean curvature vanishes identically.

Setting up. As the description of the above fact is rather intuituive, we will formulate the problem.

Let C be a simple closed smooth curve in $\mathbb{R}^3$ and set

$$\mathcal{S}_C := \left\{ f\colon \overline{D} \to \mathbb{R}^3\,;\ \begin{array}{l} f \text{ is a } C^\infty\text{-immersion} \\ f(\partial D) = C \end{array} \right\}, \tag{1.6}$$

where D (resp. $\overline{D}$) is the open (resp. closed) unit disc and ∂D is its boundary:[1]

$$(1.7)\quad \overline{D} := D \cup \partial D, \qquad \begin{aligned} D &:= \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2<1\},\\ \partial D &:= \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2=1\}\\ &= \{(\cos\theta, \sin\theta)\,;\,\theta\in\mathbb{R}\}.\end{aligned}$$

Roughly speaking, $\mathcal{S}_C$ is "the set of the surfaces bounded by C". Then we set the *area functional* as

$$(1.8)\qquad \mathcal{A}\colon \mathcal{S}_C \ni f \longmapsto \mathcal{A}(f) = \iint_{\overline{D}} |f_u \times f_v|\,du\,dv.$$

Using these notations, our result can be stated as the following:

Theorem 1.1. *If a surface $f \in \mathcal{S}_C$ attains the minimum of the area functional $\mathcal{A}$, the mean curvature of f vanishes identically.*

Taking this fact into account, we define

Definition 1.2. A surface whose mean curvature vanishes identically is said to be *minimal*.

Remark 1.3. As Theorem 1.1 is a necessary condition for the minimizer, a minimal surface is not necessarily a minimizer of the area functional.

[1] A map f defined on $\overline{D}$ is said to be C^∞ if there exists a open set $\widetilde{D}$ containing $\overline{D}$ and a C^∞ map $\tilde{f}$ defined on $\widetilde{D}$ such that $\tilde{f}|_{\overline{D}} = f$.

Variations of surfaces. To show Theorem 1.1, we want to "differentiate" the functional $\mathcal{A}$.

Definition 1.4. For a surface $f \in \mathcal{S}_C$, a *variation* (fixing the boundary) of f is a C^∞-map

$$\mathcal{F}\colon \overline{D} \times (-\varepsilon, \varepsilon) \ni (u,v;t) \longmapsto f^t(u,v) := \mathcal{F}(u,v;t) \in \mathbb{R}^3$$

such that $f^0 = f$ and $f^t \in \mathcal{S}_C$ for each $t \in (-\varepsilon,\varepsilon)$, where ε is a positive number. The vector-valued function

$$(1.9)\qquad V(u,v) := \left.\frac{\partial}{\partial t}\right|_{t=0} f^t(u,v)$$

is called the *variational vector field* of the variation $\mathcal{F}$.

Lemma 1.5. *For a variation $\mathcal{F} = \{f^t\}$ of $f \in \mathcal{S}_c$ with variational vector field V, it holds that*

$$\frac{d}{d\theta} f(\cos\theta,\sin\theta) \times V(\cos\theta, \sin\theta) = \mathbf{0}.$$

Proof. Since $(\cos\theta, \sin\theta)$ is a parametrization of ∂D, $\gamma^t(\theta) := f^t(\cos\theta,\sin\theta) \in C$ for all t and θ. Thus, two vectors in the left-hand side of the first assertion are both tangent to C, proving the lemma. □

The first variation formula.

Theorem 1.6. *Let $\mathcal{F} = \{f^t\}$ be a variation of $f \in \mathcal{S}_C$ with variational vector field V. Then it holds that*

$$(1.10)\qquad \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = -2 \iint_{\overline{D}} H\bigl(V\cdot\nu\bigr)\,dA,$$

where H, ν and dA are the mean curvature, the unit normal vector and the area element of f, respectively.

Proof. By the definition of the area (1.3), we have

$$\begin{aligned}
(*) :&= \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = \left.\frac{d}{dt}\right|_{t=0} \iint_{\overline{D}} |f_u^t \times f_v^t|\,du\,dv \\
&= \iint_{\overline{D}} \left.\frac{\partial}{\partial t}\right|_{t=0} |f_u^t \times f_v^t|\,du\,dv \\
&= \iint_{\overline{D}} \frac{\bigl(V_u \times f_v + f_u \times V_v\bigr)\cdot(f_u\times f_v)}{|f_u\times f_v|}\,du\,dv \\
&= \iint_{\overline{D}} \bigl(V_u \times f_v + f_u \times V_v\bigr)\cdot \nu\,du\,dv \\
&= \iint_{\overline{D}} \bigl(\bigl(V_u \times f_v\bigr)\cdot\nu + \bigl(f_u \times V_v\bigr)\cdot\nu\bigr)\,du\,dv.
\end{aligned}$$

Here, by the formula of *scalar triple product*

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = (\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{a} = (\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b} = \det(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}),$$

we have

$$\begin{aligned}
(*) &= \iint_{\overline{D}} \bigl(\bigl(\nu\times f_v\bigr)\cdot V_u + \bigl(f_u\times\nu\bigr)\cdot V_v\bigr)\,du\,dv \\
&= (\mathrm{II}) - (\mathrm{I}), \\
(\mathrm{I}) :&= \iint_{\overline{D}} \left[\bigl(\bigl(\nu\times f_v\bigr)\cdot V\bigr)_u + \bigl(\bigl(f_u\times\nu\bigr)\cdot V\bigr)_v\right]\,du\,dv, \\
(\mathrm{II}) :&= \iint_{\overline{D}} \left[\bigl(\bigl(\nu\times f_v\bigr)_u\cdot V\bigr) + \bigl(f_u\times\nu\bigr)_v\cdot V\bigr)\bigr)\right]\,du\,dv.
\end{aligned}$$

By the Green-Stokes formula, (I) is computed as

$$\begin{aligned}
(\mathrm{I}) &= \iint_{\overline{D}} \left[\bigl(\bigl(\nu\times f_v\bigr)\cdot V\bigr)_u - \bigl(\bigl(\nu\times f_u\bigr)\cdot V\bigr)_v\right]\,du\,dv, \\
&= \int_{\partial D} \nu\cdot\bigl(\bigl(f_u\,du + f_v\,dv\bigr)\times V\bigr) \\
&= \int_{-\pi}^{\pi} \nu\cdot\left(\frac{d}{d\theta} f(\cos\theta,\sin\theta)\times V(\cos\theta,\sin\theta)\right)d\theta = 0.
\end{aligned}$$

Here, the last assertion is obtained by Lemma 1.5. On the other hand, using the Weingarten formula (1.4), (II) is computed as

$$\begin{aligned}
(\mathrm{II}) :&= \iint_{\overline{D}} \bigl[\bigl(\nu_u\times f_v\bigr)\cdot V + \bigl(\nu\times f_{vu}\bigr)\cdot V \\
&\qquad + \bigl(f_{uv}\times\nu\bigr)\cdot V + \bigl(f_u\times\nu_v\bigr)\cdot V\bigr]\,du\,dv \\
&= \iint_{\overline{D}} \bigl[\bigl(\nu_u\times f_v\bigr)\cdot V + \bigl(f_u\times\nu_v\bigr)\cdot V\bigr]\,du\,dv \\
&= -\iint_{\overline{D}} \bigl[\bigl((A_1^1 f_u + A_1^2 f_v)\times f_v\bigr)\cdot V \\
&\qquad + \bigl(f_u\times(A_2^1 f_u + A_2^2 f_v)\bigr)\cdot V\bigr]\,du\,dv \\
&= -\iint_{\overline{D}} (A_1^1 + A_2^2)(f_u\times f_v)\cdot V\,du\,dv \\
&= -\iint_{\overline{D}} 2H(\nu\cdot V)|f_u\times f_v|\,du\,dv \qquad \square
\end{aligned}$$

Proof of Theorem 1.1. We need the following "the fundamental lemma for calculus of variations".

Lemma 1.7. *Assume a smooth function $h\colon \overline{D} \to \mathbb{R}$ satisfies*

$$\iint_{\overline{D}} h(u,v)\varphi(u,v)\,du\,dv = 0$$

for all C^∞-function with $\varphi|_{\partial D} = 0$. Then $h = 0$ on D.

Proof. Assume $h(u_0, v_0) > 0$ (resp. < 0)($(u_0, v_0) \in D$). By a continuity, there exists $\varepsilon > 0$ such that $h(u, v) > -$ on an ε-ball $B := B_\varepsilon(u_0, v_0)$ centered at (u_0, v_0). Let φ be a non-negative C^∞-function on $\overline{D}$ such that $\varphi > 0$ on B and 0 on $\overline{D} \setminus B$. Then

$$\iint_{\overline{D}} h\varphi\,du\,dv = \iint_B h\,\varphi\,du\,dv > 0 \qquad (\text{resp. } < 0),$$

a contradiction. □

Proof of Theorem 1.6. Assume $f \in \mathcal{S}_C$ minimizes the area. Then for any variation $\mathcal{F} = \{f^t\}$ of f, $\mathcal{A}(f^t)$ is not less than $\mathcal{A}(f) = \mathcal{A}(f^0)$. Then by Theorem 1.6, it holds that

$$0 = \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = -2\int_{\overline{D}} H(V\cdot\nu)|f_u \times f_v|\,du\,dv.$$

Let φ be a C^∞-function on $\overline{D}$ with $\varphi|_{\partial D} = 0$. Then $f^t := f + t\varphi\nu$ is a variation of f with variational vector field $V = \varphi\nu$. Thus,

$$\iint H|f_u \times f_v|\varphi = 0.$$

Since φ is arbitrary, Lemma 1.7 yields the conclusion. □

Exercises

1-1[H] Define a functional $\mathcal{V}\colon \mathcal{S}_C \to \mathbb{R}$ defined on $\mathcal{S}_C$ as in (1.6) as

$$\mathcal{V}(f) := \frac{1}{3}\iint_{\overline{D}} (f\cdot\nu)|f_u \times f_v|\,du\,dv \qquad (f \in \mathcal{S}_C).$$

Then

(1) Explain a geometric meaning of $\mathcal{V}(f)$.

(2) Compute $\left.\dfrac{d}{dt}\right|_{t=0} \mathcal{V}(f^t)$ for a variation $\{f^t\}$ of $f \in \mathcal{S}_C$.

Surfaces of constant mean curvature

Closed surfaces A *closed surface* in the Euclidean 3-space $\mathbb{R}^3$ is a C^∞-immersion $f\colon \Sigma \to \mathbb{R}^3$ of a compact, connected 2 dimensional manifold Σ into $\mathbb{R}^3$. Taking a local coordinate neighborhood $(U; u, v)$ of Σ, f can be identified a parametrized surface $f(u, v)$ as in the previous section.

Throughout this section, we assume that Σ is *oriented*, that is, an atlas $\{(U_\alpha; u^\alpha, v^\alpha) \mid \alpha \in A\}$ of Σ satisfying

$$\frac{\partial(u^\beta, v^\beta)}{\partial(u^\alpha, v^\alpha)} := \det J_{\alpha\beta} > 0 \qquad \text{on } U_\alpha \cap U_\beta \tag{2.1}$$

for each α, $\beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$ is specified. Here $J_{\alpha\beta}$ is the *Jacobian matrix* of the coordinate change $(u^\alpha, v^\alpha) \mapsto (u^\beta, v^\beta)$

$$J_{\alpha\beta} := \begin{pmatrix} \frac{\partial u^\beta}{\partial u^\alpha} & \frac{\partial u^\beta}{\partial v^\alpha} \\ \frac{\partial v^\beta}{\partial u^\alpha} & \frac{\partial v^\beta}{\partial v^\alpha} \end{pmatrix} \tag{2.2}$$

Fix a coordinate neighborhood $(U; u, v)$. Then the immersion $f\colon (u, v) \mapsto f(u, v)$ is considered as a vector-valued smooth function on U, and so are there derivatives f_u and f_v. Then the unit normal vector ν, the first fundamental form ds^2, the second fundamental form $I\!I$, the area element dA, the Gaussian curvature K and the mean curvature H are defined as in (1.1), (1.2), (1.3) and (1.5) in the previous section. Moreover, one can prove easily that they are independent on choice of local coordinate systems (cf. [2-1] and/or [2-2]).

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Definition 2.1. Let $f\colon \Sigma \to \mathbb{R}^3$ be an oriented closed surface. Then the *area* $\mathcal{A}(f)$ of $f(\Sigma)$ and the (signed) *volume* $\mathcal{V}(f)$ of the region bounded by $f(\Sigma)$ are defined as

$$\mathcal{A}(f) := \int_\Sigma dA, \qquad \mathcal{V}(f) := \frac{1}{3}\int_\Sigma f \cdot \nu \, dA,$$

where "$\cdot$" denotes the canonical inner product of $\mathbb{R}^3$, ν is the unit normal vector as in (1.1), and dA denotes the area element which is represented by $dA := |f_u \times f_v| \, du \, dv$ on each coordinate neighborhood $(U; u, v)$.

Remark 2.2. If the surface f is an embedding, that is, the map f is injective (in this case), the image $f(\Sigma)$ bounds a bounded and connected region D of $\mathbb{R}^3$, and the volume of D coincide with the absolute value of $\mathcal{V}(f)$.

Obviously, these two functionals have the following properties:

Lemma 2.3. *For an immersion $f \in \mathcal{S}(\Sigma)$ and a positive number $\lambda > 0$, $\mathcal{A}(\lambda f) = \lambda^2 \mathcal{A}(f)$, and $\mathcal{V}(\lambda f) = \lambda^3 \mathcal{V}(f)$ hold.*

Example 2.4 (The round sphere)**.** Let $R > 0$ be a constant and denote by

$$S^2(R) := \{\boldsymbol{x} \in \mathbb{R}^3 \mid |\boldsymbol{x}| = R\} \subset \mathbb{R}^3$$

the sphere in $\mathbb{R}^3$ of radius R centered at the origin. Then the inclusion map

$$\iota\colon S^2(R) \ni \boldsymbol{x} \longmapsto \iota(\boldsymbol{x}) = \boldsymbol{x} \in \mathbb{R}^3$$

is an embedding. A map

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (-\pi, \pi) \ni (u, v) \\ \longmapsto (R\cos u \cos v, R\cos u \sin v, R\sin u) \in S^2(R)$$

gives a local coordinate system of $S^2(R)$, and we have

$$dA = R^2 \cos u\, du\, dv, \qquad \nu = -(\cos u \cos v, \cos u \sin v, \sin u).$$

Since this coordinate neighborhood covers an open dense subset of $S^2(R)$, "integration over $S^2(R)$" is replaced by "integration over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [-\pi, \pi]$":

$$\begin{aligned} \mathcal{A}(\iota) &= \int_{-\pi/2}^{\pi/2} du \int_{-\pi}^{\pi} dv\, R^2 \cos u \\ &= 2\pi R^2 \int_{-\pi/2}^{\pi/2} \cos u\, du = 4\pi R^2, \\ \mathcal{V}(\iota) &= \frac{1}{3} \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} R^3 \cos u\, du\, dv = -\frac{4}{3}\pi R^3. \end{aligned}$$

The Gaussian and the mean curvature are computed as

$$K = \frac{1}{R^2} \qquad \text{and} \qquad H = \frac{1}{R},$$

respectively, which are constant on the surface. We call S^2_R the *round sphere* of radius R.

Area minimizing surfaces with a volume constraint. Let Σ be a compact, connected and oriented 2-manifold and consider

$$\mathcal{S}(\Sigma) = \{f \colon \Sigma \to \mathbb{R}^3 \mid f \text{ is an immersion}\}. \tag{2.3}$$

In addition, for a fixed positive constant V_0. we set

$$\mathcal{S}(\Sigma, V_0) := \{f \in \mathcal{S}(\Sigma) \mid \mathcal{V}(f) = V_0\}, \tag{2.4}$$

that is, $\mathcal{S}(\Sigma, V_0)$ is the set of immersions of Σ into $\mathbb{R}^3$ bounding given volume V_0.

In this section, we shall prove

Theorem 2.5. *If $f_0 \in \mathcal{S}(\Sigma, V_0)$ minimizes the area in $\mathcal{S}(\Sigma, V_0)$, the mean curvature of f_0 is non-zero constant.*

Theorem 2.5 and Example 2.4 give rise to the following question, known as Heinz-Hopf's problem:

Question 2.6. *Are there a closed surface of constant mean curvature which is not congruent to the round sphere?*

Variation formula for the area and the volume Similar to the previous section, we define variations of $f \in \mathcal{S}(\Sigma)$:

Definition 2.7. A *variation* of an immersion $f \colon \Sigma \to \mathbb{R}^3$ is a C^∞-map $F \colon (-\varepsilon, \varepsilon) \times \Sigma \to \mathbb{R}^3$ satisfying

- $f^t := F(t, *) \colon \Sigma \to \mathbb{R}^3$ is an immersion for each $t \in (-\varepsilon, \varepsilon)$,
- $f^0 = F(0, *)$ coincides with f.

The variational vector field V of a variation $F = \{f^t\}$ is a vector-valued function V on Σ defined by

$$V(p) := \left.\frac{\partial}{\partial t}\right|_{t=0} F(t,p) \qquad (p \in \Sigma).$$

Similar to variational formula in Section 1, we have

Theorem 2.8. *Let $\{f^t\}$ be a variation of an immersion $f\colon \Sigma \to \mathbb{R}^3$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = -2\int_\Sigma H\varphi\, dA, \qquad \left.\frac{d}{dt}\right|_{t=0} \mathcal{V}(f^t) = \int_\Sigma \varphi\, dA,$$

hold, where $\varphi := V \cdot \nu$, V is the variational vector field of $\{f^t\}$ and ν is the unit normal vector field of f.

Proof. Since almost all part of the computation in the previous section are coordinate-independent, we can show the result in a similar way to them.

Here, we shall prove the formula for the volume functional. Let $(U; u, v)$ be a local coordinate system. Then it holds that

$$\begin{aligned} \varPhi :&= f^t \cdot \nu^t |f^t_u \times f^t_v| = f^t \cdot \frac{f^t_u \times f^t_v}{|f^t_u \times f^t_v|} |f^t_u \times f^t_v| \\ &= \det(f^t, f^t_u, f^t_v) \end{aligned}$$

Differentiating this in t, we have

$$\begin{aligned} \left.\frac{\partial}{\partial t}\right|_{t=0} \varPhi &= \det(\dot f^t, f_u, f_v) + \det(f, \dot f^t_u, f_v) + \det(f, f_u, \dot f^t_v) \\ &= \det(V, f_u, f_v) + \det(f, V_u, f_v) + \det(f, f_u, V_v), \end{aligned}$$

where $\dot{*} = (\partial/\partial t)|_{t=0}$. Here, since

$$\begin{aligned} \det(V, f_u, f_v) &= V \cdot (f_u \times f_v) = (V\cdot\nu)|f_u \times f_v|, \\ \det(f, V_u, f_v) &= \bigl(\det(f, V, f_v)\bigr)_u - \det(f, V, f_{uv}) - \det(f_u, V, f_v) \\ &= \bigl(\det(f, V, f_v)\bigr)_u - \det(f, V, f_{uv}) + \det(V, f_u, f_v) \\ \det(f, f_u, V_v) &= \bigl(\det(f, f_u, V)\bigr)_v - \det(f, f_{uv}, V) - \det(f_v, f_u, V) \\ &= \bigl(\det(f, f_u, V)\bigr)_v - \det(f, f_{uv}, V) + \det(V, f_u, f_v), \end{aligned}$$

it holds that

$$\begin{aligned} \left(\left.\frac{\partial}{\partial t}\right|_{t=0} \varPhi\right) du \wedge dv &= 3(V\cdot\nu)\,|f_u \times f_v|\, du \wedge dv \\ &\quad + \Bigl(\bigl(\det(f, V, f_v)\bigr)_u + \bigl(\det(f, f_u, V)\bigr)_v\Bigr)\, du \wedge dv. \end{aligned}$$

Here, setting

$$\alpha := \det(f, V, f_u)\, du + \det(f, V, f_v)\, dv = \det(f, V, df),$$

we have the coordinate-independent expression

$$\left(\left.\frac{\partial}{\partial t}\right|_{t=0} \varPhi\right) du \wedge dv = 3(V\cdot\nu)\, dA + d\alpha,$$

and then,

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \mathcal{V}(f^t) &= \frac{1}{3}\int_\Sigma \left(\left.\frac{\partial}{\partial t}\right|_{t=0} \varPhi\right) du \wedge dv \\ &= \int_\Sigma (V\cdot\nu)\, dA + \frac{1}{3} d\alpha = \int_\Sigma (V\cdot\nu)\, dA, \end{aligned}$$

proving the formula. $\square$

Proof of Theorem 2.5. Let $f_0 \in \mathcal{S}(\Sigma, V_0)$ be an immersion minimizing area in $\mathcal{S}(\Sigma, V_0)$. Then it holds that

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = 0 \qquad \text{for any volume preserving variation } \{f^t\}. \tag{2.5}$$

Here, a variation $\{f^t\}$ of f_0 is said to be *volume preserving* if $\mathcal{V}(f^t) = \mathcal{V}(f_0)$ for all t.

Let $\{f^t\}$ be a (not necessarily volume preserving) variation of f_0. Then, by Lemma 2.3, $\{\tilde{f}^t\}$ defined by

$$\tilde{f}^t := \frac{\mathcal{V}(f^t)^{-1/3}}{\mathcal{V}(f_0)^{1/3}}\, f^t$$

is volume preserving variation, and the map $\{f^t\} \mapsto \{\tilde{f}^t\}$ is a surjection to the set of volume preserving variations. That is, (2.5) is equivalent to

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}\left(\frac{\mathcal{V}(f^t)^{-1/3}}{\mathcal{V}(f_0)^{-1/3}}\, f^t\right) = 0 \quad \text{for any variation } \{f^t\}. \tag{2.6}$$

Here, by Theorem 2.8,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}\bigl(\mathcal{V}(f^t)^{-1/3}\, f^t\bigr) &= \left.\frac{d}{dt}\right|_{t=0} \mathcal{V}(f^t)^{-2/3}\mathcal{A}(f^t) \\
&= -\frac{2}{3}\dot{\mathcal{V}}(f^t)\mathcal{V}(f_0)^{-5/3}\mathcal{A}(f_0) + \mathcal{V}(f_0)^{-2/3}\dot{\mathcal{A}}(f^t) \\
&= \mathcal{V}(f_0)^{-2/3}\left(-\frac{2}{3}\frac{\mathcal{A}(f_0)}{\mathcal{V}(f_0)}\dot{\mathcal{V}}(f^t) + \dot{\mathcal{A}}(f^t)\right) \\
&= \mathcal{V}(f_0)^{-2/3}\left(\int_\Sigma \left(-\frac{2}{3}\frac{\mathcal{A}(f_0)}{\mathcal{V}(f_0)} - 2H\right)\varphi\, dA\right),
\end{aligned}$$

where $\dot{*} = (d/dt)|_{t=0}$ and $\varphi = V \cdot \nu$. Then by Lemma 1.7,

$$-\frac{2}{3}\frac{\mathcal{A}(f_0)}{\mathcal{V}(f_0)} - 2H = 0,$$

holds, and then H is constant.

References

[2-1] 梅原雅顕，山田光太郎，曲線と曲面（改訂版），裳華房，2014.

[2-2] Masaaki Umehara and Kotaro Yamada, Differential Geometry of Curves and Surfaces, (trasl. by Wayne Rossman), World Scientific, 2017.

Exercises

2-1[H] Let $\mathcal{C} := \{\gamma\colon S^1 \to \mathbb{R}^2 \mid \gamma' \neq \mathbf{0}\}$ be the set of regular closed curves on $\mathbb{R}^2$.

(1) Define the area $\mathcal{A}(\gamma)$ of the region bounded by γ.

(2) Let $\mathcal{C}(a)$ be the set of curves γ with $\mathcal{A}(\gamma) = a$. Show that if a curve $\gamma_0 \in \mathcal{C}(a)$ minimizes the length in $\mathcal{C}(a)$, the curvature of γ_0 is constant.

Hint: A curve $\gamma \in \mathcal{C}(a)$ can be parametrized $\gamma(t) = {}^t(x(t), y(t))$ as a 2π-periodic function. The length $\mathcal{L}(\gamma)$ and the curvature function κ of γ are defined as

$$\mathcal{L}(\gamma) := \int_0^{2\pi} |\dot{\gamma}(t)|\, dt, \qquad \kappa(t) := \frac{\det(\dot{\gamma}(t), \ddot{\gamma}(t))}{|\dot{\gamma}(t)|^3}$$

where $\dot{} = d/dt$.

Examples of Constant Mean Curvature Surfaces

Planar curves. Let $\gamma\colon I \ni s \mapsto \gamma(s) \in \mathbb{R}^2$ be a smooth map defined on an $I \subset \mathbb{R}$. Then γ is called a *regular curve* if $\dot\gamma \neq 0$ on I, where $\dot{} = d/ds$. The parameter s is called an *arc length parameter* if

$$|\dot\gamma(s)| = \left|\frac{d\gamma}{ds}(s)\right| = 1 \tag{3.1}$$

holds on I.

Lemma 3.1. *A regular curve $\gamma\colon I \ni t \mapsto \gamma(t) \in \mathbb{R}^2$ defined on an interval $I \subset \mathbb{R}$ can be reparametrized by an arc length parameter. Moreover, such an arc length parameter is unique up to additive constants.*

Proof. Fix $t_0 \in I$ and define a function $s\colon I \to \mathbb{R}$ by

$$s(t) := \int_{t_0}^t \left|\frac{d\gamma}{dt}(u)\right| du.$$

Then $s\colon I \to J \subset \mathbb{R}$ is a smooth function such that $ds/dt > 0$. Hence there exists the smooth inverse $J \ni s \mapsto t(s) \in I$. Then $\tilde\gamma(s) := \gamma\bigl(t(s)\bigr)$ is the desired reparametrization. In fact,

$$\begin{aligned}\left|\frac{d\tilde\gamma(s)}{ds}\right| &= \left|\frac{d\gamma}{dt}\bigl(t(s)\bigr)\frac{dt}{ds}(s)\right| = \left|\frac{d\gamma}{dt}\bigl(t(s)\bigr)\frac{1}{ds/dt(t(s))}\right| \\ &= \left|\frac{d\gamma}{dt}\bigl(t(s)\bigr)\frac{1}{|d\gamma/dt\bigl(t(s)\bigr)|}\right| = 1.\end{aligned}$$

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So we have the first assertion. Let s and u be two arc length parameters. Then there exists a parameter change $u = u(s)$, which is strictly increasing function such that

$$1 = \left|\frac{d\gamma}{ds}\right| == \left|\frac{d\gamma}{du}\frac{du}{ds}\right| = \frac{du}{ds}\left|\frac{d\gamma}{du}\right| = \frac{du}{ds}.$$

Hence $u = s +$ constant, proves the second assertion. □

Throughout this section, we assume that planar curves are parameterized by arc length parameter.

Let $\gamma(s) = {}^t\bigl(x(s), y(s)\bigr)$ $(s \in I)$be a parametrized planar curve where s is an arc length parameter. Then

$$\boldsymbol{e}(s) := \dot\gamma(s) = \begin{pmatrix}\dot x(s)\\ \dot y(s)\end{pmatrix}, \qquad \boldsymbol{n}(s) := \begin{pmatrix}-\dot y(s)\\ \dot x(s)\end{pmatrix}$$

are mutually perpendicular orthogonal vectors for each $s \in I$. Thus we have obtained a map

$$\mathcal{F}(s) := \bigl(\boldsymbol{e}(s), \boldsymbol{n}(s)\bigr)\colon I \longmapsto \mathrm{SO}(2), \tag{3.2}$$

where $\mathrm{SO}(2)$ is the set (a group) of 2×2-orthogonal matrix of determinant 1. We call $\mathcal{F}$ the *frame* of γ. Note that

$$\mathrm{SO}(2) = \{R(\theta) \,|\, \theta \in \mathbb{R}\}, \quad R(\theta) := \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}. \tag{3.3}$$

Theorem 3.2 (The Frenet formula)**.** *Let $\mathcal{F}(s)$ be the frame of the curve $\gamma(s)$ where s is an arc length parameter defined on an* *interval I. Then there exists a unique smooth function $\kappa\colon I \to \mathbb{R}$ such that*

$$\dot{\mathcal{F}} = \mathcal{F}\varOmega \qquad \varOmega(s) := \kappa(s)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.4}$$

Proof. Since $\mathcal{F}$ is a function valued on $\mathrm{SO}(2)$, $\mathcal{F}^{-1}\dot{\mathcal{F}}$ is valued on the set of skew-symmetric matrices. In fact, since ${}^t\mathcal{F} = \mathcal{F}^{-1}$,

$$\begin{aligned} {}^t\left(\mathcal{F}^{-1}\dot{F}\right) &= {}^t\left({}^t\mathcal{F}\dot{\mathcal{F}}\right) = {}^t\dot{\mathcal{F}}\mathcal{F} = \frac{d}{ds}\mathcal{F}^{-1}\mathcal{F} \\ &= -\mathcal{F}^{-1}\dot{\mathcal{F}}F^{-1}\mathcal{F} = -\mathcal{F}^{-1}\dot{\mathcal{F}}. \end{aligned}$$

Hence there exists a function $\kappa(s)$ such that

$$\mathcal{F}^{-1}\dot{\mathcal{F}} = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix},$$

proving the theorem. □

We call the function κ the *curvature* of the curve γ.

Proposition 3.3. *Let $\gamma(s) = {}^t\big(x(s), y(s)\big)$ be a planar curve parametrized by the arc length s. Then its curvature satisfies*

$$\kappa = \dot{x}\ddot{y} - \dot{y}\ddot{x}.$$

Theorem 3.4 (The fundamental theorem for planar curves)**.** *Let $\kappa\colon I \to \mathbb{R}$ be a smooth function. Then there exists a curve $\gamma\colon I \to \mathbb{R}$ parametrized by the arc length whose curvature is κ. Moreover, such a curve γ is unique up to rotations and translations of $\mathbb{R}^2$.*

Proof. First we shall prove uniqueness: Let γ_j $(j = 1, 2)$ be curves with curvature κ, and denote by $\mathcal{F}_j$ $(j = 1, 2)$ the frame of γ_j. Then by (3.4),

$$\begin{aligned} \frac{d}{ds}(\mathcal{F}_2\mathcal{F}_1^{-1}) &= \frac{d}{ds}(\mathcal{F}_2{}^t\mathcal{F}_1) == \dot{\mathcal{F}}_2{}^t\mathcal{F}_1 + \mathcal{F}_2{}^t\dot{\mathcal{F}}_1 \\ &= \mathcal{F}_2\varOmega{}^t\mathcal{F}_1 + \mathcal{F}_2{}^t\mathcal{F}_1\varOmega = \mathcal{F}_2(\varOmega + {}^t\varOmega){}^t\mathcal{F}_1 = O \end{aligned}$$

holds, and thus there exist constant matrix such that

$$\mathcal{F}_2\mathcal{F}_1^{-1} = A \qquad (A \in \mathrm{SO}(2)),$$

that is, $\mathcal{F}_2 = A\mathcal{F}_1$. Comparing the first column of this, we have

$$\dot{\gamma}_2 = A\dot{\gamma}_1 \qquad \text{and then} \qquad \gamma_2 = A\dot{\gamma}_1 + \boldsymbol{a},$$

where $A \in \mathrm{SO}(2)$ and $\boldsymbol{a} \in \mathbb{R}^2$. Hence the uniqueness part holds.

Next, we prove existence: fix $s_0 \in I$ and set

$$\gamma(s) := \int_{s_0}^{s}\left(\cos\int_{s_0}^{u}\kappa(t)\,dt, \sin\int_{s_0}^{u}\kappa(t)\,dt\right)du.$$

Then one can check that s is the arc length parameter of $\gamma(s)$, and $\kappa(s)$ is the curvature. □

Surfaces of revolution. Let $\gamma(s) = \big(x(s), y(s)\big)$ be a regular curve parametrized by the arc length s, satisfying $y(s) > 0$ for all s. Then the *surface of revolution* of γ about the x-axis is parametrized as

$$f(t, s) := \big(x(s), y(s)\cos t, y(s)\sin t\big), d \qquad (t, s) \in S^1 \times I. \tag{3.5}$$

The curve γ is called the *profile curve* of the surface (3.5).

Noticing $\dot{x}^2 + \dot{y}^2 = 1$, the first fundamental form I and the second fundamental form of f are expressed as

$$I = y^2\, dt^2 + ds^2, \quad II = -\dot{x}y\, dt^2 + (\dot{x}\ddot{y} - \dot{y}\ddot{x})\, ds^2 = -\dot{x}y\, dt^2 + \kappa\, ds^2,$$

where κ is the curvature of the profile curve (cf. Proposition 3.3). Hence we have

Proposition 3.5. *The mean curvature function H of the surface* (3.5) *is expressed as*

$$2H = \kappa - \frac{\dot{x}}{y}. \tag{3.6}$$

Delaunay surfaces.

Theorem 3.6. *Let H be a non-zero constant. Then the profile curve $(x(s), y(s))$ of a surface of revolution with constant mean curvature H is expressed as*

$$\begin{aligned} y(s) &= \frac{1}{2H}\sqrt{(2Ha+1)^2 - 2(2Ha+1)\cos 2Hs}, \\ x(s) &= \int_0^s \frac{1 + (2aH+1)\cos 2Hu}{y(u)}\, du, \end{aligned} \tag{3.7}$$

up to horizontal translations and parameter changes, where a is a constant.

Proof. Let $\gamma(s) := \bigl(x(s), y(s)\bigr)$ be the profile curve of given surface of revolution with constant mean curvature H. Then by (3.6), the curvature function κ of γ satisfies

$$\kappa = 2H + \frac{\dot{x}}{y}.$$

Thus, the frame $\mathcal{F}$ of γ satisfies the Frenet formula (Theorem 3.2):

$$\dot{\mathcal{F}} = \left(2H + \frac{\dot{x}}{y}\right)\mathcal{F}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.8}$$

We shall find the curve solving this differential equation. Set

$$\widetilde{\mathcal{F}} := y\mathcal{F}.$$

Then, noticing

$$\dot{x}^2 + \dot{y}^2 = 1, \tag{3.9}$$

the equation (3.8) is equivalent to

$$\dot{\widetilde{\mathcal{F}}} = 2H\mathcal{F}\begin{pmatrix} 0 & -2H \\ 2H & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.10}$$

Let

$$\begin{aligned} A(s) &:= \widetilde{\mathcal{F}}(s)\mathcal{F}_0(s)^{-1}, \\ \mathcal{F}_0(s) &:= R(2Hs) = \begin{pmatrix} \cos 2Hs & -\sin 2Hs \\ \sin 2Hs & \cos 2Hs \end{pmatrix}. \end{aligned} \tag{3.11}$$

Substituting $\widetilde{\mathcal{F}} = A\mathcal{F}_0$ into (3.10), we have

$$\dot{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathcal{F}_0^{-1} = \begin{pmatrix} \sin 2Hs & -\cos 2Hs \\ \cos 2Hs & \sin 2Hs \end{pmatrix}, \tag{3.12}$$

and then

$$A = \frac{-1}{2H}\begin{pmatrix} \cos 2Hs & \sin 2Hs \\ -\sin 2Hs & \cos 2Hs \end{pmatrix} + C, \tag{3.13}$$

where C is a constant matrix. Summing up, it holds that

$$\begin{aligned} y\mathcal{F} = \widetilde{\mathcal{F}} &= A\mathcal{F}_0 \\ &= \frac{1}{2H}\left(C\begin{pmatrix} \cos 2Hs & -\sin 2Hs \\ \sin 2Hs & \cos 2Hs \end{pmatrix} - \mathrm{id}\right). \end{aligned} \tag{3.14}$$

Since right-hand side is a periodic function and $\mathcal{F} \in \mathrm{SO}(2)$, y^2 (and then y) is a periodic function. Hence y must take both maximum and minimum. By a change of parameter s to $s+$ constant and a horizontal translation $x \mapsto x+$constant, we may assume y takes its maximum at $s = 0$, and $x(0) = 0$. Moreover, by the reflection of the y-axis, we may assume $\dot{x}(0) \geqq 0$ without loss of generality. Hence we can assume an initial condition

$$\bigl(x(0), y(0)\bigr) = (0, a), \quad \bigl(\dot{x}(0), \dot{y}(0)\bigr) = (1, 0), \quad \ddot{y}(0) = \kappa(0) \leqq 0.$$

Substituting these into (3.14), we have $C = (2Ha + 1)\,\mathrm{id}$:

$$y\mathcal{F} = \frac{1}{2H}\left((2Ha+1)\begin{pmatrix} \cos 2Hs & -\sin 2Hs \\ \sin 2Hs & \cos 2Hs \end{pmatrix} - \mathrm{id}\right). \tag{3.15}$$

Taking the determinant of this, we have

$$y^2 = \frac{1}{(2H)^2}\bigl((2Ha+1)\cos 2Hs - 1)^2 + (2Ha^2+1)\sin^2 2Hs\bigr)$$

and then

$$y = \frac{1}{2H}\sqrt{(2Ha+1)^2 - 2(2Ha+1)\cos 2Hs}.$$

On the other hand, the $(1,1)$-component of (**??**) is expressed as

$$y\dot{x} = \frac{1}{2H}\bigl(1 + (2aH+1)\cos 2Hs\bigr).$$

Thus we have the conclusion. $\square$

The surfaces in (3.7) are called the *Delaunay surfaces*.

References

[3-1] 剱持勝衛：「曲面論講義 — 平均曲率一定曲面入門」（培風館，2000）.

[3-2] K. Kenmotsu, Surfaces with constant mean curvature, Translations of Mathematical Monographs, translated by Katsuhiro Moriya, American Math. Soc., 2003.

Exercises

3-1[H] Draw pictures of Delaunay curves for $H = \frac{1}{2}$.

3-2[H] Classify minimal surfaces of revolution.