## 2017 2Q

Wireless Communication Engineering

## \#7 Detection and <br> Error due to Noise

Kei Sakaguchi sakaguchi@mobile.ee.

July 3, 2017

## Course Schedule (1)

|  | Date | Text | Contents |
| :--- | :--- | :--- | :--- |
| \#1 | June 12 | 1,7 | Introduction to wireless communication systems |
| \#2 | June 15 | 2,5, etc | Link budget design of wireless access |
| \#3 | June 19 |  | Up/down conversion and equivalent baseband <br> system |
| \#4 | June 22 | $3.3,3.4$ | Digital modulation and pulse shaping |
| \#5 | June 26 | 3.5 | Demodulation and matched filter |
| \#6 | June 29 |  | Collaborative exercise for better understanding 1 |
| \#7 | July 3 | 3.5 | Detection and error due to noise |
| \#8 | July 6 | 4.4 | Channel fading and diversity combining |

## From Previous Lecture

■ Analog demodulation

$$
\begin{gathered}
y(t)=y_{\mathrm{BI}}(t) \cos 2 \pi f_{0} t-y_{\mathrm{BQ}}(t) \sin 2 \pi f_{0} t \\
y_{\mathrm{BI}}(t)=2 \int y(t) \cos 2 \pi f_{0} t \mathrm{~d} t, \quad y_{\mathrm{BQ}}(t)=-2 \int y(t) \sin 2 \pi f_{0} t \mathrm{~d} t
\end{gathered}
$$

■ Matched filter

$$
\begin{gathered}
y_{\mathrm{D}}(t)=g_{\mathrm{r}}(t) \otimes y_{\mathrm{B}}(t)=g_{\mathrm{r}}(t) \otimes g_{\mathrm{s}}(t) \otimes s_{\mathrm{D}}(t)+g_{\mathrm{r}}(t) \otimes n_{\mathrm{B}}(t) \\
G_{\mathrm{r}}(f)=\left(G_{\mathrm{s}}(f)\right)^{*}, \quad g_{\mathrm{r}}(t)=g_{\mathrm{s}}(-t) \\
\gamma=\frac{|g(0)|^{2}}{\mathrm{E}\left[|n(t)|^{2}\right]} \leq \frac{1}{N_{0}} \int\left|G_{\mathrm{s}}(f)\right|^{2} \mathrm{~d} f=\frac{E_{\mathrm{s}}}{N_{0}}=\frac{P_{\mathrm{s}} T_{s}}{N_{0}}=\frac{P_{\mathrm{s}}}{N_{0} B}=\frac{P_{\mathrm{s}}}{\sigma^{2}}
\end{gathered}
$$

## Contents

- Coherent detection
- Error rate of BPSK signal
- Error rate of QPSK signal
- Error rate of QAM signal
- Demonstration


## Transmitter \& Receiver

Transmitter


## QAM Modulation


$\log _{2} M=1$

$\log _{2} M=4$

$\log _{2} M=2$

$\log _{2} M=6$

## Gray Coding (Mapping)



## Average Power with Fixed $E_{0}$

$$
\begin{aligned}
& P_{2}=E_{0} \quad \longrightarrow{ }_{\sqrt{E_{0}}}^{\longrightarrow} \quad P_{4}=2 E_{0}
\end{aligned}
$$

$$
\begin{aligned}
& \text { 64QAM }
\end{aligned}
$$

## Power Normalization

Average power with fixed $E_{0}$

$$
\begin{aligned}
\tilde{P}_{M} & =2\left(\frac{2}{\sqrt{M}} \sum_{i=1}^{\sqrt{M} / 2}(2 i-1)^{2} E_{0}\right) \\
& =\frac{2(M-1) E_{0}}{3} \quad(M \geq 4)
\end{aligned}
$$

Power normalization

$$
\begin{array}{rlr}
\tilde{P}_{M}=P_{s} \longrightarrow E_{0} & =\frac{3}{2(M-1)} P_{s} & (M \geq 4) \\
E_{0} & =P_{s} & (M=2)
\end{array}
$$

## Output of Matched Filter



Output of matched filter

$$
y_{\mathrm{D}}(t)=g(t) \otimes h_{\mathrm{B}} s_{\mathrm{D}}(t)+g_{\mathrm{r}}(t) \otimes n_{\mathrm{B}}(t)
$$



Design matched filter as

$$
\begin{aligned}
& g(0)=\int G(f) \mathrm{d} f=\int G_{\mathrm{r}}(f) G_{\mathrm{s}}(f) \mathrm{d} f=1, \quad \int\left|G_{\mathrm{r}}(f)\right|^{2} \mathrm{~d} f=1 \\
& y_{\mathrm{D}}(k)=h_{\mathrm{B}} s_{\mathrm{D}}(k)+n_{\mathrm{D}}(k), \quad \mathrm{E}\left[\left|n_{\mathrm{D}}(k)\right|^{2}\right]=\mathrm{E}\left[\left|n_{\mathrm{B}}(k)\right|^{2}\right]=N_{0} B=\sigma^{2}
\end{aligned}
$$

## Detection Schemes

| Detection | Modulation | Demodulation |
| :---: | :---: | :---: |
| Envelope | ASK | $\hat{s}_{\mathrm{D}}(k)=\left\|y_{\mathrm{D}}(k)\right\| /\left\|h_{\mathrm{B}}\right\|$ |
| Correlation | FSK | $\left\|\int y_{\mathrm{B}}(t) \exp (j \pi \Delta f t) \mathrm{d} t\right\| \geq \leq \int y_{\mathrm{B}}(t) \exp (-j \pi \Delta f t) \mathrm{d} t \mid$ |
| Differential | Differential mod. <br> $\tilde{\theta}(k)=\theta(k)+\theta(k-1)$ | $\hat{s}_{\mathrm{D}}(k)=y_{\mathrm{D}}(k) / y_{\mathrm{D}}(k-1)$ |
| Coherent | PSK, QAM, MSK | $\hat{s}_{\mathrm{D}}(t)=y_{\mathrm{D}}(t) / h_{\mathrm{B}}$ |

## Coherent Detection



Output of matched filter

$$
\begin{aligned}
& y_{\mathrm{D}}(k)=h_{\mathrm{B}} s_{\mathrm{D}}(k)+n_{\mathrm{D}}(k) \\
& \Rightarrow y(k)=h_{\mathrm{B}} s(k)+n(k)
\end{aligned}
$$

Coherent detection

$$
\hat{s}(k)=y(k) / \hat{h}_{\mathrm{B}}
$$

$\qquad$ Compensation of channel response
Digital demodulation


$$
\hat{m}(k)=f^{-1}(\hat{s}(k))
$$

## Channel Estimation

Output of matched filter

$$
y(k)=h_{\mathrm{B}} s_{\mathrm{TR}}(k)+n(k)
$$



Frame structure of transmit signal

$$
\text { \# of training symbols: } K
$$



## Output of Coherent Detection

Output of coherent detection

$$
\begin{aligned}
\hat{s}(k) & =y(k) / h_{\mathrm{B}} \\
& =s(k)+n(k) / h_{\mathrm{B}}
\end{aligned}
$$



Complex Gaussian with variance $\sigma^{2} /\left|h_{B}\right|^{2}$

Output of coherent detection

$$
p\left(\hat{s}_{\mathrm{I}}\right)= \begin{cases}\frac{1}{\sqrt{\pi \sigma^{2} /\left|h_{\mathrm{B}}\right|^{2}}} \exp \left(-\frac{\left(\hat{s}_{\mathrm{I}}-\sqrt{E_{0}}\right)^{2}}{\sigma^{2} /\left|h_{\mathrm{B}}\right|^{2}}\right), & m=0 \\ \frac{1}{\sqrt{\pi \sigma^{2} /\left|h_{\mathrm{B}}\right|^{2}}} \exp \left(-\frac{\left(\hat{s}_{\mathrm{I}}+\sqrt{E_{0}}\right)^{2}}{\sigma^{2} /\left|h_{\mathrm{B}}\right|^{2}}\right), & m=1\end{cases}
$$

## Error Rate of BPSK Signal

Pairwise error rate

$$
p_{\mathrm{pw}}=\frac{1}{\sqrt{\pi \sigma^{2} /\left|h_{\mathrm{B}}\right|^{2}}} \int_{0}^{\infty} \exp \left(-\frac{\left(\hat{s}_{\mathrm{I}}+\sqrt{E_{0}}\right)^{2}}{\sigma^{2} /\left|h_{\mathrm{B}}\right|^{2}}\right) \mathrm{d} \hat{s}_{\mathrm{I}} \xrightarrow[{-\sqrt{E_{0}}}]{0 \sqrt{E_{0}}} \underset{\hat{s}_{\mathrm{I}}(k)}{\longrightarrow}
$$

$$
=\frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_{0}\left|h_{\mathrm{B}}\right|^{2}}{\sigma^{2}}}\right)
$$

Error rate of BPSK signal
Complementary error function

$$
\operatorname{erfc}(x)=\frac{2}{\sqrt{\pi}} \int_{x}^{\infty} \exp \left(-z^{2}\right) \mathrm{d} z
$$

$$
E_{0}=P_{s}
$$

$$
\gamma=\frac{P_{s}\left|h_{\mathrm{B}}\right|^{2}}{\sigma^{2}}
$$

## Receive SNR

$$
p_{\mathrm{eb}}=\frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{P_{s}\left|h_{\mathrm{B}}\right|^{2}}{\sigma^{2}}}\right)=\frac{1}{2} \operatorname{erfc}(\sqrt{\gamma})
$$

## Error Rate of QPSK Signal

Output of coherent detection

$$
\hat{s}(k)=s_{\mathrm{I}}(k)+j s_{\mathrm{Q}}(k)+n(k) / h_{\mathrm{B}}
$$

Bit error rate

$$
\begin{aligned}
& p_{\mathrm{pw}}=\frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_{0}\left|h_{\mathrm{B}}\right|^{2}}{\sigma^{2}}}\right) \\
& E_{0}=P_{\mathrm{s}} / 2 \\
& p_{\mathrm{eb}}=\frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{P_{\mathrm{s}}\left|h_{\mathrm{B}}\right|^{2}}{2 \sigma^{2}}}\right)=\frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{\gamma}{2}}\right)
\end{aligned}
$$



Symbol error rate

$$
p_{\mathrm{es}}=1-\frac{\left(1-p_{\mathrm{eb}}\right)}{\mathrm{l}} \frac{\left(1-p_{\mathrm{eb}}\right)}{\mathrm{Q}}=2 p_{\mathrm{cb}}-p_{\mathrm{cb}}^{2} \cong 2 p_{\mathrm{cb}}
$$

## Error Rate of QAM Signal

Symbol error rate

$$
\begin{aligned}
& p_{\text {es }}=1-\left(1-p_{\text {esI }}\right)\left(1-p_{\text {esQ }}\right) \cong 2 p_{\text {esI }}=2 p_{\text {esQ }} \\
& p_{\text {esI }}=\frac{\left(\frac{\sqrt{M}-2}{\sqrt{M}}\right) \times 2 p_{\mathrm{pw}}+\left(\frac{2}{\sqrt{M}}\right) \times p_{\mathrm{pw}}}{\text { Center }} \frac{\text { Two edges }}{\text { Tw }} \\
& =\left(1-\frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{E_{0}\left|h_{\mathrm{B}}\right|^{2}}{\sigma^{2}}}\right)
\end{aligned}
$$

## Error Rate of QAM Signal

Symbol error rate

$$
p_{\mathrm{es}}=2\left(1-\frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{E_{0}\left|h_{\mathrm{B}}\right|^{2}}{\sigma^{2}}}\right)
$$

Bit error rate

$$
\begin{aligned}
E_{0} & =\frac{3}{2(M-1)} P_{s} \\
p_{\mathrm{eb}} & \cong \frac{1}{\log _{2} M} p_{\mathrm{es}} \quad \begin{array}{l}
\text { Symbol error corresponds to } \\
\text { one bit error owing to Gray coding }
\end{array} \\
& =\frac{2}{\log _{2} M}\left(1-\frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{3 \gamma}{2(M-1)}}\right)
\end{aligned}
$$

## Error Rate Performance

Bit error rate performance


## Summary

- Channel estimation \& coherent detection

$$
\hat{h}_{\mathrm{B}}=\frac{1}{K} \sum_{k=1}^{K} y(k) / s_{\mathrm{TR}}(k) \longrightarrow \hat{s}(k)=y(k) / h_{\mathrm{B}}=s(k)+n(k) / h_{\mathrm{B}}
$$

■ Error rate of BPSK signal

$$
p_{\mathrm{eb}}=\frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{P_{\mathrm{s}}\left|h_{\mathrm{B}}\right|^{2}}{\sigma^{2}}}\right)=\frac{1}{2} \operatorname{erfc}(\sqrt{\gamma})
$$

- Error rate of QAM signal

$$
p_{\mathrm{eb}}=\frac{2}{\log _{2} M}\left(1-\frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{3 \gamma}{2(M-1)}}\right)
$$

## Demo



## Error Rate of MSK

Output of coherent detection

$$
\hat{s}(k)=s(k)+n(k) / h_{\mathrm{B}}
$$

MSK modulation

$$
\begin{aligned}
& s(k)=\exp (j \theta(k)) \\
& \theta(k)=\pi a_{k} / 2+\theta(k-1)
\end{aligned}
$$

BER of MSK signal

$$
p_{\mathrm{eb}}=\frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{P_{s}\left|h_{\mathrm{B}}\right|^{2}}{\sigma^{2}}}\right)=\frac{1}{2} \operatorname{erfc}(\sqrt{\gamma})
$$

Constellation



