9. Scattering Matrix

The matrix representations of circuit are addressed focusing on the scattering matrix.

The topics included:

9.1 Definition of scattering matrix

9.2 Scattering matrix of loss-less circuit

9.3 Relation between scattering matrix and other matrices

9.4 Method for determination of scattering matrix

9.1 Definition

Along a transmission line

$$V(y) = V_i e^{j\beta y} + V_r e^{-j\beta y} = \sqrt{R_c}\left(\qquad\qquad + \qquad\qquad \right)$$

$$I(y) = \frac{1}{R_c}\left(V_i e^{j\beta y} - V_r e^{-j\beta y}\right) = \frac{1}{\sqrt{R_c}}\left(\qquad\qquad - \qquad\qquad \right)$$

Let's define the following quantity.

$$a(y) = \frac{V_i}{\sqrt{R_c}} e^{j\beta y} \rightarrow |a(y)|^2 = \frac{|V_i|^2}{R_c} : \text{incident power}$$

$$b(y) = \frac{V_r}{\sqrt{R_c}} e^{-j\beta y} \rightarrow |b(y)|^2 = \frac{|V_r|^2}{R_c} : \text{reflected power}$$

$$V(y) = \sqrt{R_c}\left(a(y) + b(y)\right)$$

$$I(y) = \frac{1}{\sqrt{R_c}}\left(a(y) - b(y)\right)$$

$\Rightarrow$

$$a(y) = \frac{V(y) + R_c I(y)}{2\sqrt{R_c}}$$

$$b(y) = \frac{V(y) - R_c I(y)}{2\sqrt{R_c}}$$

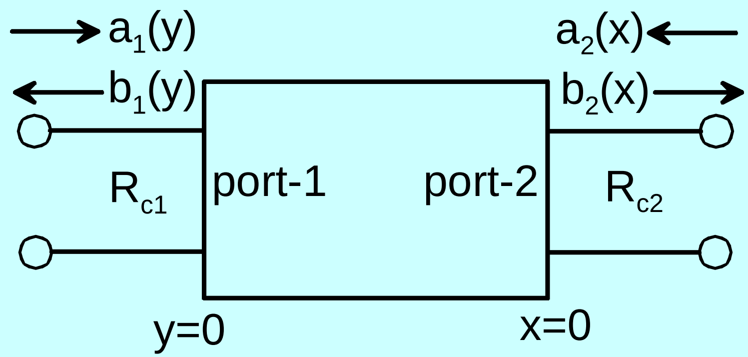
2-port circuit

$$a_1(y) = \frac{V_1(y) + R_{c1} I_1(y)}{2\sqrt{R_{c1}}}$$

$$b_1(y) = \frac{V_1(y) - R_{c1} I_1(y)}{2\sqrt{R_{c1}}}$$

$$a_2(x) = \frac{V_2(x) + R_{c2} I_2(x)}{2\sqrt{R_{c2}}}$$

$$b_2(x) = \frac{V_2(x) - R_{c2} I_2(x)}{2\sqrt{R_{c2}}}$$



Definition of scattering matrix: $a_1 = a_1(0)$, etc

$$b_1 = S_{11} a_1 + S_{12} a_2$$

$$b_2 = S_{21} a_1 + S_{22} a_2$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\mathbf{b} = \mathbf{Sa}$$

9.2 Loss-less properties

Input power is given by

$$|a_1|^2 + |a_2|^2 + \ldots + |a_n|^2 = \qquad\qquad = \tilde{\mathbf{a}}\mathbf{a}$$

$\tilde{\mathbf{a}}$: transpose conjugate of $\mathbf{a}$

Loss-less : input power = output power

$$\tilde{\mathbf{a}}\mathbf{a} = \tilde{\mathbf{b}}\mathbf{b}$$

$$\mathbf{b} = \mathbf{S}\mathbf{a}$$

$$\tilde{\mathbf{b}} =$$

$$\tilde{\mathbf{a}}\mathbf{a} - \tilde{\mathbf{b}}\mathbf{b} = \tilde{\mathbf{a}}\mathbf{a} - \qquad\qquad = \tilde{\mathbf{a}} \qquad\qquad \mathbf{a} = 0$$

$$\therefore \tilde{\mathbf{S}}\mathbf{S} =$$

Reciprocity:
$$\left.\frac{b_2}{a_1}\right|_{a2=0} = \left.\frac{b_1}{a_2}\right|_{a1=0}$$

$$\therefore S_{21} = S_{12}$$

$$S_{ij} = S_{ji}$$

9.3 S-matrix and T-matrix

T-matrix

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} b_2 \\ a_2 \end{pmatrix}$$

suitable for cascade connection

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \mathbf{T}_1 \begin{pmatrix} b_2' \\ a_2' \end{pmatrix}$$

$$\begin{pmatrix} b_2' \\ a_2' \end{pmatrix} = \mathbf{T}_2 \begin{pmatrix} b_2 \\ a_2 \end{pmatrix} \qquad \therefore \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \qquad\qquad\qquad \begin{pmatrix} b_2 \\ a_2 \end{pmatrix}$$

from the definition of S-matrix

$$a_1 = \frac{1}{S_{21}} b_2 - \frac{S_{22}}{S_{21}} a_2$$

$$b_1 = \frac{S_{11}}{S_{21}} b_2 + \left(S_{12} - \frac{S_{11} S_{22}}{S_{21}} \right) a_2$$

$$T_{11} = \frac{1}{S_{21}}, \qquad T_{12} = -\frac{S_{22}}{S_{21}}$$

$$T_{21} = \frac{S_{11}}{S_{21}}, \qquad T_{22} = S_{12} - \frac{S_{11} S_{22}}{S_{21}}$$

S-matrix and Z-matrix

$$a_j = \frac{V_j + R_{cj} I_j}{2\sqrt{R_{cj}}}$$

$$b_j = \frac{V_j - R_{cj} I_j}{2\sqrt{R_{cj}}}$$

$$\Rightarrow \quad \boxed{V_j} = \sqrt{R_{cj}}(a_j + b_j)$$

$$\boxed{I_j} = \frac{1}{\sqrt{R_{cj}}}(a_j - b_j)$$

Then,

$$\boxed{\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix}} = \begin{pmatrix} \sqrt{R_{c1}} & & & 0 \\ & \sqrt{R_{c2}} & & \\ & & \ddots & \\ 0 & & & \sqrt{R_{cn}} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} \sqrt{R_{c1}} & & & 0 \\ & \sqrt{R_{c2}} & & \\ & & \ddots & \\ 0 & & & \sqrt{R_{cn}} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\boxed{[V]} = [\sqrt{R}]([a] + [b])$$

$$[\sqrt{R}] = \begin{pmatrix} \sqrt{R_{c1}} & & & 0 \\ & \sqrt{R_{c2}} & & \\ & & \ddots & \\ 0 & & & \sqrt{R_{cn}} \end{pmatrix}$$

S-matrix and Z-matrix

Similarly,

$$[I] = [\sqrt{R}]^{-1}([a] - [b])$$

$$[\sqrt{R}]^{-1} = \begin{pmatrix} \sqrt{R_{c1}} & & & 0 \\ & \sqrt{R_{c2}} & & \\ & & \ddots & \\ 0 & & & \sqrt{R_{cn}} \end{pmatrix}^{-1} = \begin{pmatrix} \sqrt{R_{c1}}^{-1} & & & 0 \\ & \sqrt{R_{c2}}^{-1} & & \\ & & \ddots & \\ 0 & & & \sqrt{R_{cn}}^{-1} \end{pmatrix}$$

Since $[V] = [Z][I]$

$$[V] = [V_i] + [V_r]$$

$$[I] = [R]^{-1}([V_i] - [V_r])$$

$$[V_i] + [V_r] = [Z][R]^{-1}([V_i] - [V_r])$$

$$(1 + [Z][R]^{-1})[V_r] = ([Z][R]^{-1} - 1)[V_i]$$

S-matrix and Z-matrix

$$(1+[Z][R]^{-1})[V_r] = ([Z][R]^{-1}-1)[V_i]$$

$$[V_r] = [\sqrt{R}][b] \qquad [V_i] = [\sqrt{R}][a]$$

$$[b] = [S][a]$$

Thus, $(1+[Z][R]^{-1})[\sqrt{R}][S] = ([Z][R]^{-1}-1)[\sqrt{R}]$

$$\therefore [S] = [\sqrt{R}]^{-1}(1+[Z][R]^{-1})^{-1}([Z][R]^{-1}-1)[\sqrt{R}]$$

Especially, when $R_{c\,j} = R_0 \quad (\mathrm{j} = 1,2,3\ldots\mathrm{n})$

$$[\sqrt{R}]^{-1} = \qquad = \frac{1}{\sqrt{R_0}}[1]$$

$$[R]^{-1} = \frac{1}{R_0}[1], \qquad [\sqrt{R}] = \sqrt{R_0}[1]$$

$$[S] = \left(1+\frac{1}{\mathrm{R}_0}[Z]\right)^{-1}\left(\frac{1}{\mathrm{R}_0}[Z]-1\right) = ([\mathrm{Z}]+\mathrm{R}_0)^{-1}([\mathrm{Z}]-\mathrm{R}_0)$$

9.4 How to find S-matrix elements ?

Simple method:

$$b_1 = S_{11}a_1 \rightarrow S_{11} = \left.\frac{b_1}{a_1}\right|_{a2=0}$$

$$b_2 = S_{21}a_1 \rightarrow S_{21} = \left.\frac{b_2}{a_1}\right|_{a2=0}$$

, but $a_2=0$ is hard to realize.

Deshamp's method

Terminate port-2 with an impedance Z_{L2}

→ Reflection by the impedance Z_{L2} is given by $S_2 = \frac{a_2}{b_2} = \frac{Z_{L2} - R_{c2}}{Z_{L2} + R_{c2}}$

$$\frac{b_2}{a_1} = \frac{b_2}{a_2}\frac{a_2}{a_1} = \frac{1}{S_2}\frac{a_2}{a_1} = S_{21} + S_{22}\frac{a_2}{a_1}$$

$$\rightarrow \frac{a_2}{a_1}\left(\frac{1}{S_2} - S_{22}\right) = S_{21}, \qquad \frac{a_2}{a_1} = \frac{S_{21}S_2}{1 - S_{22}S_2}$$

Reflection at port-1: $S_1 = \frac{b_1}{a_1} = S_{11} + S_{12}\frac{a_2}{a_1} = \frac{S_{11} + (S_{12}S_{21} - S_{11}S_{22})S_2}{1 - S_{22}S_2}$

How to find S-matrix elements ?

S_1 and S_2 : linear transformation

$$(1 - S_{22}S_2)S_1 = S_{11} + (S_{12}S_{21} - S_{11}S_{22})S_2$$

$$\left((S_{12}S_{21} - S_{11}S_{22}) + S_{22}S_1\right)S_2 = S_1 - S_{11}$$

$$S_2 = \frac{S_1 - S_{11}}{S_{22}S_1 + (S_{12}S_{21} - S_{11}S_{22})}$$

When S_2 moves along a circle $S_2 = \exp(-2j\beta x)$
S_1 moves also along another circle.

$|S_2| = 1$ (S_2 is located on a unit circle) gives the following relation.

$$\left(1 - S_{22}\overline{S_{22}}\right)S_1\overline{S_1} - \left((S_{12}S_{21} - S_{11}S_{22})\overline{S_{22}} + S_{11}\right)\overline{S_1} - \left(\overline{(S_{12}S_{21} - S_{11}S_{22})}S_{22} + \overline{S_{11}}\right)S_1 + \left(S_{11}\overline{S_{11}} - (S_{12}S_{21} - S_{11}S_{22})\overline{(S_{12}S_{21} - S_{11}S_{22})}\right) = 0$$

$$\text{center} : \frac{D\overline{S_{22}} + S_{11}}{1 - S_{22}\overline{S_{22}}} = \frac{S_{12}S_{21}\overline{S_{22}}}{1 - S_{22}\overline{S_{22}}} + S_{11}$$

$$\text{radius} : \frac{\sqrt{\left|D\overline{S_{22}} + S_{11}\right|^2 - (1 - S_{22}\overline{S_{22}})(S_{11}\overline{S_{11}} - D\overline{D})}}{1 - S_{22}\overline{S_{22}}} = \frac{|S_{12}S_{21}|}{1 - S_{22}\overline{S_{22}}}$$

$$\text{where} \quad D = (S_{12}S_{21} - S_{11}S_{22})$$

How to find S-matrix elements ?

By changing the position of movable short, x, we can measure the corresponding reflection S_1.

(${S_1}$ moves also along another circle.)

e.g.

for $S_2 = \pm 1$ $$S_1 = \frac{S_{11} \pm (S_{12}S_{21} - S_{11}S_{22})}{1 \mp S_{22}}$$

By the correspondence between S_1 and S_2, we can determine S_{ij}.