5.1 Rectangular waveguide

EM field of TE and TM mode

orthogonal relation among eigen modes

Consider TE mode in the following.

$$\gamma E_y = -j\omega\mu H_x \qquad (5.1.a)$$

$$-\gamma E_x = -j\omega\mu H_y \qquad (5.1.b)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \qquad (5.1.c)$$

$$\frac{\partial H_z}{\partial y} + \gamma H_y = j\omega\varepsilon E_x \qquad (5.1.d)$$

$$-\gamma H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \qquad (5.1.e)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \qquad (5.1.f)$$

$$E_x = \frac{-j\omega\mu}{\gamma^2 + \omega^2\varepsilon\mu}\frac{\partial H_z}{\partial y} = \frac{-j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial y}$$

$$E_y = \frac{j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial x}$$

$$H_x = \frac{-\gamma}{\beta_c^2}\frac{\partial H_z}{\partial x}$$

$$H_y = \frac{-\gamma}{\beta_c^2}\frac{\partial H_z}{\partial y}$$

Rectangular waveguide: TE mode

$$\frac{j\omega\mu}{\beta_c^2}\frac{\partial^2 H_z}{\partial x^2}+\frac{j\omega\mu}{\beta_c^2}\frac{\partial^2 H_z}{\partial y^2}=-j\omega\mu H_z$$

$$\therefore \frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}=-\beta_c^2 H_z$$

$$\frac{1}{X(x)}\frac{d^2X}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y}{dy^2}=-\beta_c^2$$

$$\frac{1}{X(x)}\frac{d^2X}{dx^2}=-\beta_x^2$$

$$\frac{1}{Y(y)}\frac{d^2Y}{dy^2}=-\beta_y^2$$

$$\beta_x^2+\beta_y^2=\beta_c^2$$

$$X(x)=A\sin\beta_x x+B\cos\beta_x x$$

$$Y(y)=C\sin\beta_y y+D\cos\beta_y y$$

Rectangular waveguide: TE mode

$$H_z = X(x)Y(y)$$

$$X(x) = A\sin\beta_x x + B\cos\beta_x x$$

$$Y(y) = C\sin\beta_y y + D\cos\beta_y y$$

boundary condition at x=0, a

$$E_y = \frac{j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial x} = \frac{j\omega\mu}{\beta_c^2}\beta_x(A\cos\beta_x x - B\sin\beta_x x)Y(y)$$

$$A = 0 \qquad (E_y = 0 \quad at \quad x = 0)$$

$$\sin\beta_x a = 0 \qquad (E_y = 0 \quad at \quad x = a)$$

$$\therefore \beta_x = \frac{m\pi}{a}$$

boundary condition for E_x at y=0,b

$$C = 0$$

$$\sin\beta_y b = 0 \qquad \therefore \beta_y = \frac{n\pi}{b}$$

Rectangular waveguide: TE mode

Field components are determined as

$$H_z = H_0 \cos\frac{m\pi}{a}x \cos\frac{n\pi}{b}y$$

(Either n or m can be 0)

$$E_x = \frac{j\omega\mu}{\beta_c^2} H_0 \frac{n\pi}{b} \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y$$

$$E_y = \frac{-j\omega\mu}{\beta_c^2} H_0 \frac{m\pi}{a} \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y$$

$$H_x = \frac{\gamma}{\beta_c^2} H_0 \frac{m\pi}{a} \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y$$

$$H_y = \frac{\gamma}{\beta_c^2} H_0 \frac{n\pi}{b} \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y$$

$$\beta_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \gamma^2 + \omega^2\varepsilon\mu$$

$$\therefore \gamma = \pm\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\varepsilon\mu}$$

Rectangular waveguide: TM mode

TM mode

$$E_z = E_0 \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \qquad \boxed{(m,n \geq 1)}$$

$$E_x = \frac{-\gamma}{\beta_c^2} E_0 \frac{m\pi}{a} \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y$$

$$E_y = \frac{-\gamma}{\beta_c^2} E_0 \frac{n\pi}{b} \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y$$

$$H_x = \frac{j\omega\varepsilon}{\beta_c^2} E_0 \frac{n\pi}{b} \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y$$

$$H_y = \frac{-j\omega\varepsilon}{\beta_c^2} E_0 \frac{m\pi}{a} \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y$$

$$\gamma = \pm\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\varepsilon\mu}$$

Rectangular waveguide: orthogonality

TE mode (between transverse field component)

$$E_t^e = \frac{j\omega\mu}{\beta_c^2}\left(-\frac{\partial H_z}{\partial y} i + \frac{\partial H_z}{\partial x} j\right)$$

$$H_t^e = \frac{-\gamma}{\beta_c^2}\left(\frac{\partial H_z}{\partial x} i + \frac{\partial H_z}{\partial y} j\right)$$

$$\therefore E_t^e \cdot H_t^e = \frac{-j\omega\mu\gamma}{\beta_c^4}\left(-\frac{\partial H_z}{\partial x}\frac{\partial H_z}{\partial y} + \frac{\partial H_z}{\partial x}\frac{\partial H_z}{\partial y}\right) = 0$$

TM mode (between transverse field component)

$$E_t^m = \frac{-\gamma}{\beta_c^2}\left(\frac{\partial E_z}{\partial x} i + \frac{\partial E_z}{\partial y} j\right)$$

$$H_t^m = \frac{j\omega\varepsilon}{\beta_c^2}\left(\frac{\partial E_z}{\partial y} i - \frac{\partial E_z}{\partial x} j\right)$$

$$\therefore E_t^m \cdot H_t^m = \frac{-j\omega\varepsilon\gamma}{\beta_c^4}\left(\frac{\partial E_z}{\partial x}\frac{\partial E_z}{\partial y} - \frac{\partial E_z}{\partial x}\frac{\partial E_z}{\partial y}\right) = 0$$

5.2 Circular waveguide

$$\nabla \times H = i_r\left(\frac{1}{r}\frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z}\right) + i_\theta\left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right) + i_z\frac{1}{r}\left(\frac{\partial (rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)$$

TE mode

$$H_z = H_0 J_n(\beta_c r)\cos n\theta$$

$$E_r = \frac{-j\omega\mu}{\beta_c^2}\frac{1}{r}\frac{\partial H_z}{\partial \theta}$$

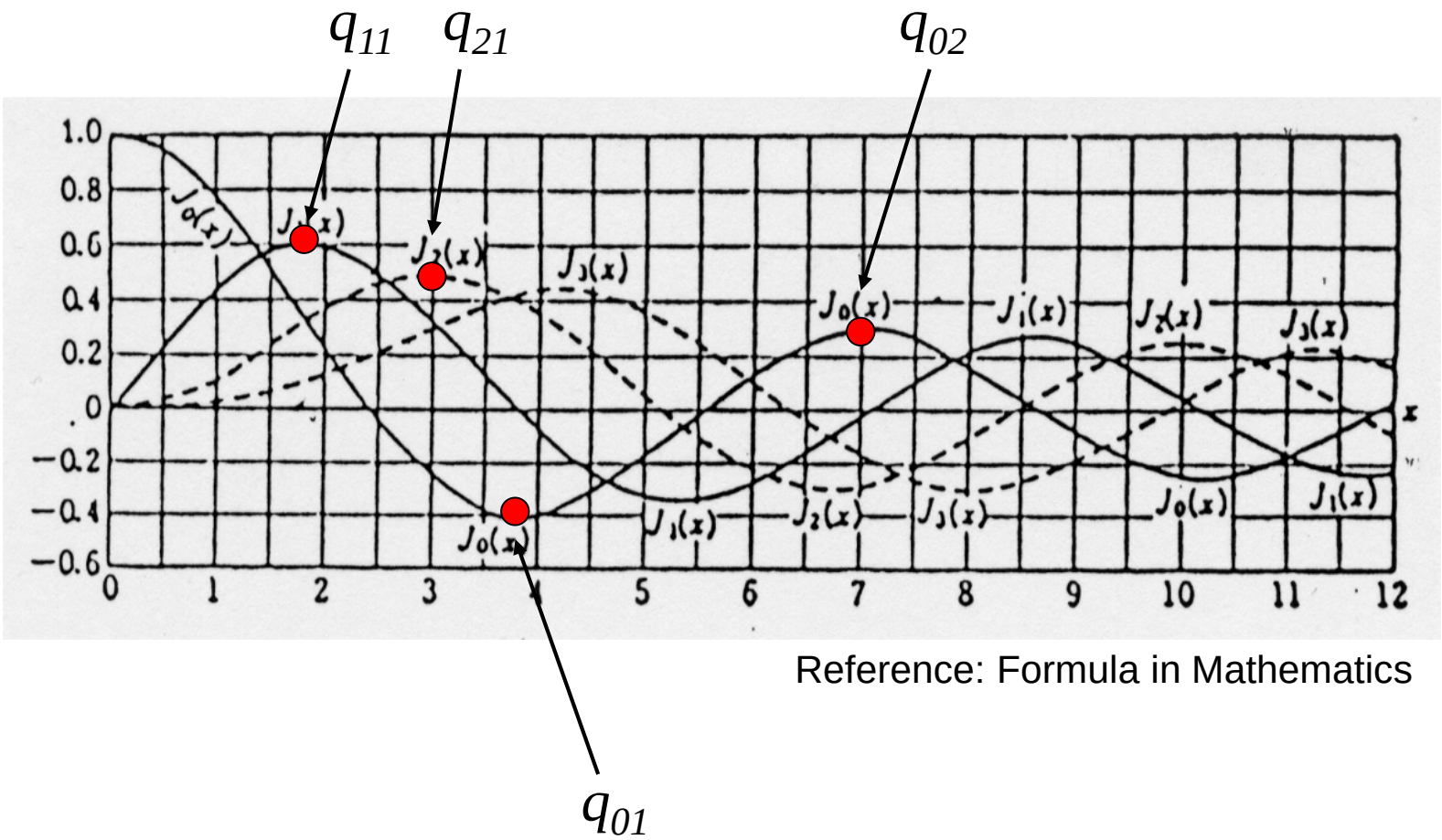
$$E_\theta = \frac{j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial r} \longrightarrow E_\theta = \frac{j\omega\mu}{\beta_c^2}\beta_c H_0 J_n'(\beta_c r)\cos n\theta$$

$$H_r = \frac{-\gamma}{\beta_c^2}\frac{\partial H_z}{\partial r}$$

$$H_\theta = \frac{-\gamma}{\beta_c^2}\frac{1}{r}\frac{\partial H_z}{\partial \theta} \qquad (\beta_c^2 = \gamma^2 + \omega^2\varepsilon\mu)$$

$$J_n'(\beta_c a) = 0 \rightarrow q_{nm} = \beta_c a \quad \therefore \beta_c = \frac{q_{nm}}{a}, \quad \gamma = \pm\sqrt{\left(\frac{q_{nm}}{a}\right)^2 - \omega^2\varepsilon\mu}$$

Bessel function $J_n(x)$



Reference: Formula in Mathematics

Circular waveguide : TM mode

TM mode

$$E_z = E_0 J_n(\beta_c r)\cos n\theta$$

$$E_r = \frac{-\gamma}{\beta_c^2}\frac{\partial E_z}{\partial r}$$

$$E_\theta = \frac{-\gamma}{\beta_c^2}\frac{1}{r}\frac{\partial E_z}{\partial \theta} \longrightarrow E_\theta = \frac{\gamma}{\beta_c^2}\frac{n}{r}E_0 J_n(\beta_c r)\sin n\theta$$

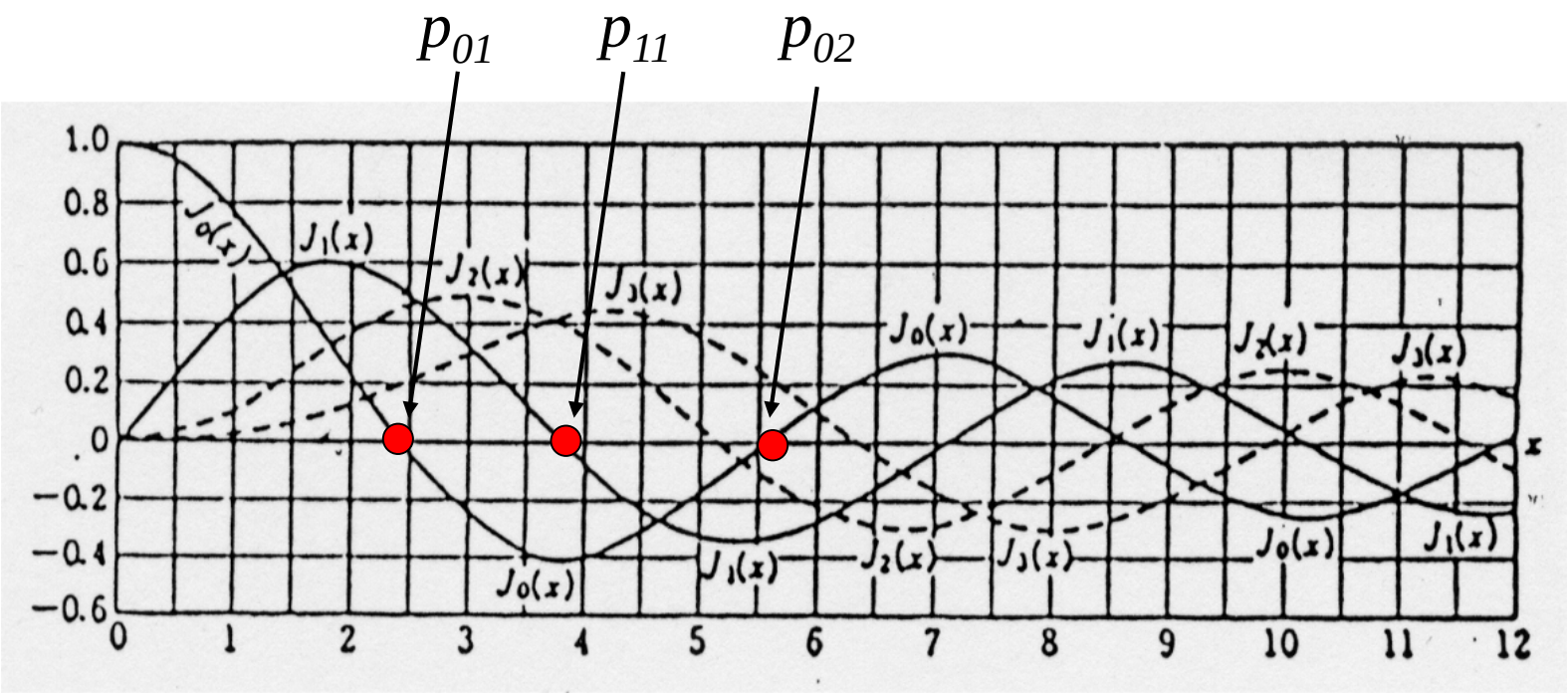
$$H_r = \frac{j\omega\varepsilon}{\beta_c^2}\frac{1}{r}\frac{\partial E_z}{\partial \theta}$$

$$H_\theta = \frac{-j\omega\varepsilon}{\beta_c^2}\frac{\partial E_z}{\partial r}$$

$$(\beta_c^2 = \gamma^2 + \omega^2\varepsilon\mu)$$

$$J_n(\beta_c a) = 0 \rightarrow p_{nm} = \beta_c a \quad \therefore \beta_c = \frac{p_{nm}}{a}, \quad \gamma = \pm\sqrt{\left(\frac{p_{nm}}{a}\right)^2 - \omega^2\varepsilon\mu}$$

Bessel function $J_n(x)$



Reference: Formula in Mathematics

5.3 Cut-off

cut-off frequency:

- The frequency at which the propagation constant of the mode concerned γ becomes 0.
- The mode is evanescent below that frequency.

Rectangular ($a = 2b$): listed in the order of lower cut-off frequency

TE_{10}

TE_{20}, TE_{01} (same cut-off frequency)

TE_{11}, TM_{11}

TE_{21}, TM_{21} .....

$$\gamma = \pm\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\varepsilon\mu}$$

Circular : listed in the order of lower cut-off frequency

TE_{11}

TM_{01}

TE_{21}

TE_{01}, TM_{11}, ...

$$\text{TE}-\text{mode}: \quad \gamma = \pm\sqrt{\left(\frac{q_{nm}}{a}\right)^2 - \omega^2\varepsilon\mu}$$

$$\text{TM}-\text{mode}: \quad \gamma = \pm\sqrt{\left(\frac{p_{nm}}{a}\right)^2 - \omega^2\varepsilon\mu}$$