Neutron Transport Theory Lecture Note (6)
- One-speed diffusion theory of a nuclear reactor (2) -

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5.2 The critical condition for general bare geometries

Considering a bare reactor of uniform composition surrounded by a free surface characterized by vacuum boundary conditions.

If the reactor is critical then the neutron flux must satisfy the steady-state diffusion equation.

$$-D\nabla^2\phi + \Sigma_a\phi(\mathbf{r}) = \nu\Sigma_f\phi(\mathbf{r}) \qquad \cdots (1)$$

boundary condition : $\phi(\tilde{\mathbf{r}}_s) = 0$ $\quad \tilde{\mathbf{r}}_s$: extrapolated boundary

Dividing Eq.(1) by $-D$,

$$\nabla^2\phi + \left(\frac{\nu\Sigma_f - \Sigma_a}{D}\right)\phi(\mathbf{r}) = 0 \qquad \cdots (2)$$

boundary condition : $\phi(\tilde{\mathbf{r}}_s) = 0$

$$\text{i.e.} \quad \nabla^2\phi + \left(\frac{k_\infty - 1}{L^2}\right)\phi(\mathbf{r}) = 0 \qquad \cdots (3)$$

boundary condition : $\phi(\tilde{\mathbf{r}}_s) = 0$

This equation is identical to that which generates the special eigenfunctions for this geometry.

$$\nabla^2\psi_n + {B_n}^2\psi_n(\mathbf{r}) = 0 \qquad \cdots (4)$$

boundary condition : $\psi(\tilde{\mathbf{r}}_S) = 0$

The requirement that the reactor is critical is the same as that for slab reactor,

$$ {B_m}^2 \equiv \left(\frac{\nu\Sigma_f - \Sigma_a}{D}\right) = {B_1}^2 \equiv {B_g}^2 \qquad \cdots (5)$$

The critical neutron flux distribution $\phi(\mathbf{r})$ is given by the fundamental eigenfunction $\psi_1(\mathbf{r})$.

Geometric buckling and flux profile for various bare core

Bare core geometry	Geometric buckling	Flux profile
Slab (thickness : a)	$\left(\frac{\pi}{\tilde{a}}\right)^2$	$\cos\frac{\pi x}{\tilde{a}}$
Sphere (radius : R)	$\left(\frac{\pi}{\tilde{R}}\right)^2$	$r^{-1}\sin\left(\frac{\pi r}{\tilde{R}}\right)$
Rectangular parallelepiped	$\left(\frac{\pi}{\tilde{a}}\right)^2+\left(\frac{\pi}{\tilde{b}}\right)^2+\left(\frac{\pi}{\tilde{c}}\right)^2$	$\cos\left(\frac{\pi x}{\tilde{a}}\right)\cos\left(\frac{\pi x}{\tilde{b}}\right)\cos\left(\frac{\pi x}{\tilde{c}}\right)$

Eq.(4) will provide us with flux shape only in critical reactor.

The magnitude of neutron flux shall be determined by the total power P generated by the core.

$$P = \int_V d^3r w_f \Sigma_f \phi(\mathbf{r}) \qquad \cdots (6)$$

w_f : energy produced per fission event

5.3 Reflected reactor geometries

We consider a slab reactor with reflectors of nonmultiplying material of thickness of to the both side of the core.

Time-independent diffusion equation (x≥0)

$$\text{Core}: \ -D^C\frac{d^2\phi^C}{dx^2}+\left(\Sigma_a^C-\nu\Sigma_f^C\right)\phi^C(x)=0, \qquad 0\le x\le\frac{a}{2} \qquad \cdots (7)$$

$$\text{Reflector}: \ -D^R\frac{d^2\phi^R}{dx^2}+\Sigma_a^R\phi^R(x)=0, \qquad \frac{a}{2}\le x\le\frac{a}{2}+\tilde{b} \qquad \cdots (8)$$

Boundary conditions

(a) $\phi^C\left(\frac{a}{2}\right)=\phi^R\left(\frac{a}{2}\right)$

(b) $J^C\left(\frac{a}{2}\right)=J^R\left(\frac{a}{2}\right)$ $\cdots (9)$

(c) $\phi^R\left(\frac{a}{2}+\tilde{b}\right)=0$

General solution in the core（symmetric solution）

$$\phi^C(x) = A^C \cos B_m^C x$$

$$\text{where, } {B_m^C}^2 \equiv \frac{\nu\Sigma_f^C - \Sigma_a^C}{D^C} \qquad \cdots (10)$$

Solution in reflector which satisfies boundary condition (c)

$$\phi^R(x) = A^R \sinh\left[\frac{\frac{a}{2} + \tilde{b} - x}{L^R}\right] \qquad \cdots (11)$$

$$\text{where, } L^R = \sqrt{\frac{D^R}{\Sigma_a^R}}$$

By applying interface boundary conditions (a) and (b),

$$A^C \cos\left(\frac{B_m^C a}{2}\right) = A^R \sinh\left(\frac{\tilde{b}}{L^R}\right) \qquad \cdots (12)$$

$$D^C B_m^C A_c \sin\left(\frac{B_m^C a}{2}\right) = \frac{D^R}{L^R} A^R \cosh\left(\frac{\tilde{b}}{L^R}\right) \qquad \cdots (13)$$

Dividing Eq.(13) by Eq.(12),

$$D^C B_m^C \tan\left(\frac{B_m^C a}{2}\right) = \frac{D^R}{L^R} \coth\left(\frac{\tilde{b}}{L^R}\right) \qquad \cdots (14)$$

This equation is the reactor critical condition.

（cf. ${B_m}^2 = {B_g}^2$ in bare core）

Rewrite Eq.(14) as

$$\left(\frac{B_m^C a}{2}\right) \tan\left(\frac{B_m^c a}{2}\right) = \frac{D^R a}{2D^C L^R} \coth\left(\frac{\tilde{b}}{L^R}\right) \qquad \cdots (15)$$

$$\frac{B_m^C a}{2} < \frac{\pi}{2} \quad \text{or} \quad {B_m^C}^2 < \left(\frac{\pi}{a}\right)^2$$

$$\left[\begin{array}{c}\text{In bare (unreflected) core} \\ {B_m}^2 = \left(\frac{\pi}{\tilde{a}}\right)^2\end{array}\right]$$

It is conventional to define the difference between bare and reflected core dimensions as the reflector savings δ :

$$\delta = [a(\text{bare}) - a(\text{reflected})]/2 \qquad \cdots(16)$$

Ex. The reflector savings for the slab core

$$\delta = \frac{1}{B_m^C}\tan^{-1}\left[\frac{D^C B_m^C L^R}{D^R}\tanh\left(\frac{\tilde{b}}{L^R}\right)\right] \qquad \cdots(17)$$

For the thick reflector $b \gg L^R$

$$\delta \cong \frac{D^C}{D^R}L^R \qquad \cdots(18)$$

5.4 Reactor criticality calculations

(1) General procedure to determine geometries and material composition of critical reactors

Diffusion equation

$$-\nabla D\nabla\phi + \Sigma_a\phi(\mathbf{r}) = \nu\Sigma_f\phi(\mathbf{r}) \qquad \cdots(19)$$

(no solutions in general unless the reactor is critical)

boundary condition

$$\phi(\tilde{\mathbf{r}}_s) = 0$$

We introduce on arbitrary parameter "k" into the equation.

$$-\nabla D\nabla\phi + \Sigma_a\phi(\mathbf{r}) = \frac{1}{k}\nu\Sigma_f\phi(\mathbf{r}) \qquad \cdots(20)$$

Picking up a core size and composition and solve the equation while determining k. (eigenvalue problem)

k : multiplication eigenvalue

(2) Solution of eigenvalue problem by power method

Rewriting Eq.(20) in operator notation

$$M\phi = \frac{1}{k}F\phi \qquad \cdots(21)$$

where, $M \equiv -\nabla D(\mathbf{r})\nabla + \Sigma_a(\mathbf{r}) \equiv$ Destruction operator (leakage plus absorption)

$F \equiv \nu\Sigma_f(\mathbf{r}) \equiv$ Production operator (fission)

Assuming the estimate $\phi^{(n)}$ and $k^{(n)}$are given.
Estimate of fission source

$$S^{(n)} = F\phi^{(n)} \qquad \cdots(22)$$

We can iteratively solve for an improved source estimates $S^{(n+1)}$ from an earlier estimate $S^{(n)}$ by solving

$$M\phi^{(n+1)} = \frac{1}{k^{(n)}} S^{(n)} \qquad \cdots(23)$$

for $\phi^{(n+1)}$ and then computing

$$S^{(n+1)} = F\phi^{(n+1)} \qquad \cdots(24)$$

as n becomes large, $\phi^{(n+1)}$ will converge to the true eigenfunction $\phi(\mathbf{r})$ that satisfies Eq.(21) with the eigenvalue

$$M\phi^{(n+1)} \cong \frac{1}{k^{(n+1)}} F\phi^{(n+1)} \qquad \cdots(25)$$

If we integrate Eq.(25) overall space, we should be able to obtain a resonance estimate for $k^{(n+1)}$ as

$$k^{(n+1)} \cong \frac{\int d^3rF\phi^{(n+1)}}{\int d^3rM\phi^{(n+1)}} \qquad \cdots(26)$$

From Eq.(23), Eq.(24)

$$k^{(n+1)} \cong \frac{\int d^3rS^{(n+1)}(\mathbf{r})}{\underline{\frac{1}{k^{(n)}} \int d^3rS^{(n)}(\mathbf{r})}} \qquad \cdots(27)$$

effective fission sources

that generate $S^{(n+1)}$

This shows the eigenvalue, k in Eq.(21) is the same as the effective multiplication factor, that is the ratio of the number of neutrons in two consecutive fission generations in the reactor.

In k≠1, to make the reactor critical, we can change the reactor size and composition and repeat the calculation.