

Neutron Transport Theory Lecture Note (5)
- One-speed diffusion theory of a nuclear reactor (1) -

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5. One-speed diffusion theory of a nuclear reactor

5.1 The time-dependent "slab" reactor

(a) Solution of diffusion equation

Considering a uniform slab of fissile material characterized by cross sections

$\Sigma_a, \Sigma_{tr}, \Sigma_f$ (Slab reactor)

One-speed diffusion equation

$$\frac{1}{v} \frac{\partial \phi}{\partial t} - D \frac{\partial^2 \phi}{\partial x^2} + \Sigma_a \phi(x, t) = v \Sigma_f \phi(x, t) \quad \dots (1)$$

Initial condition

$$\phi(x, 0) = \phi_0(x) = \phi_0(-x) \quad (\text{symetric}) \quad \dots (2)$$

Boundary conditions

$$\phi\left(\frac{\tilde{a}}{2}, t\right) = \phi\left(-\frac{\tilde{a}}{2}, t\right) = 0 \quad \dots (3)$$

A solution of the form (separation variables)

$$\phi(x, t) = \psi(x)T(t) \quad \dots (4)$$

Substituting Eq.(4) to Eq.(1) and dividing by $\psi(x)T(t)$

$$\frac{1}{T} \frac{dT}{dt} = \frac{v}{\psi} \left[D \frac{d^2 \psi}{dx^2} + (v \Sigma_f - \Sigma_a) \psi(x) \right] = \text{constant} \equiv -\lambda \quad \dots (5)$$

hence

$$\frac{dT}{dt} = -\lambda T(t) \quad \dots (6)$$

$$D \frac{d^2 \psi}{dx^2} + (v \Sigma_f - \Sigma_a) \psi(x) = -\frac{\lambda}{v} \psi(x) \quad \dots (7)$$

Solution of the time-dependent Eq.(6)

$$T(t) = T(0)e^{-\lambda t} \quad \dots (8)$$

Space dependent equation

$$D \frac{d^2 \psi}{dx^2} + \left(\frac{\lambda}{v} + v \Sigma_f - \Sigma_a \right) \psi(x) = 0 \quad \dots (9)$$

Boundary condition

$$\psi\left(\frac{\tilde{a}}{2}\right) = \psi\left(-\frac{\tilde{a}}{2}\right) = 0 \quad \dots (10)$$

here λ is still to be determined.

Considering the eigenvalue problem.

$$\begin{aligned} & \frac{d^2 \psi}{dx^2} + B_m^2 \psi_n(x) \\ = 0 & \quad \dots (11) \\ \psi_n\left(\frac{\tilde{a}}{2}\right) = \psi_n\left(-\frac{\tilde{a}}{2}\right) = 0 & \end{aligned}$$

We are interested in symmetric solutions since $\phi_0(x)$ is symmetric.

eigen functions : $\psi_n(x) = \cos B_n x$

$$\text{eigenvalue : } B_n^2 = \left(\frac{n\pi}{\tilde{a}}\right)^2, \quad n = 1, 3, 5, \dots \quad \dots (12)$$

If we identify Eq.(9) as the same problem, we must choose

$$\begin{aligned} \lambda &= v \Sigma_a + v D B_n^2 - v \Sigma_f \equiv \lambda_n, \quad n = 1, 3, 5 \\ \lambda_n &: \text{time eigenvalues} \end{aligned} \quad \dots (13)$$

General solution of Eq.(1),

$$\phi(x, t) = \sum_{\substack{n \\ \text{odd}}} A_n \exp(-\lambda_n t) \cos \frac{n\pi x}{\tilde{a}} \quad \dots (14)$$

The solution satisfies the boundary conditions. From initial condition Eq.(2),

$$\phi(x, 0) = \phi_0(x) = \sum_{\substack{n \\ \text{odd}}} A_n \cos \frac{n\pi x}{\tilde{a}} \quad \dots (15)$$

Using orthogonality,

$$A_n = \frac{2}{\tilde{a}} \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx \phi_0(x) \cos \frac{n\pi x}{\tilde{a}} \quad \dots (16)$$

Thus

$$\phi(x, t) = \sum_{\substack{n \\ \text{odd}}} \left[\frac{2}{\tilde{a}} \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx' \phi_0(x') \cos B_n x' \right] \exp(-\lambda_n t) \cdot \cos B_n x \quad \dots (17)$$

where the time eigenvalues λ_n are given by

$$\lambda_n = \nu \Sigma_a + \nu D B_n^2 - \nu \Sigma_f, \quad B_n = \frac{n\pi}{\tilde{a}} \quad \dots (18)$$

(b) Long time behavior

From Eq.(12)

$$B_1^2 < B_3^2 < \dots < B_n^2 = \left(\frac{n\pi}{\tilde{a}} \right)^2 \quad \dots (19)$$

hence from Eq.(18)

$$\lambda_1 < \lambda_3 < \lambda_5 \dots \quad \dots (20)$$

This means that the modes (terms in Eq.(17)) corresponding to larger n decay out rapidly in time.

as $t \rightarrow \infty$

$$\phi(x, t) \sim A_1 \exp(-\lambda_1 t) \cos B_1 x \quad \dots (21)$$

(fundamental mode)

This shows that regardless of the initial shape $\phi_0(x)$ the flux will decay into the fundamental mode shape.

It is usual to refer the value of B_1^2 characterizing this model as

$$B_1^2 = \left(\frac{\pi}{\tilde{a}} \right)^2 \equiv B_g^2 \equiv \text{geometric buckling} \quad \dots (22)$$

$$\left[\begin{array}{l} \text{Thus nomenclature is used since } B_n^2 \text{ is a} \\ \text{measure of the curvature of the mode shape} \\ B_n^2 = -\frac{1}{\psi_n} \frac{d^2 \psi_n}{dx^2} \end{array} \right]$$

(c) Criticality condition

What is required to make the flux distribution in the reactor time-independent

i.e. what is required to make the fission chain reaction steady-state

We will define this situation to be that of reactor criticality :

Criticality \equiv

when a time-independent neutron flux can be sustained in the reactor

(in the absence of sources other than fissions)

The general solution of the flux

$$\phi(x,t) = A_1 \exp(-\lambda_1 t) \cos B_1 x + \sum_{\substack{n=3 \\ \text{odd}}}^{\infty} A_n \exp(-\lambda_n t) \cos B_n x \quad \dots (23)$$

It is evident that requirement for a time-independent flux is just that the fundamental eigenvalue vanish.

$$\lambda_1 = 0 = \nu(\Sigma_a - \nu\Sigma_f) + \nu D B_1^2 \quad \dots (24)$$

since then higher modes ($n=3,5,\dots$) will have negative $-\lambda_n$ and decay out in time, leaving just,

$$\phi(x,t) \rightarrow A_1 \cos B_1 x \neq \text{function of time}$$

From Eq.(24), using notation $B_1^2 = B_g^2$

$$B_m^2 = B_g^2 \quad (\text{criticality condition}) \quad \dots (25)$$

$$\text{where, } B_m^2 \equiv \frac{\nu\Sigma_f - \Sigma_a}{D} \quad (\text{material buckling}) \quad \dots (26)$$

To achieve a critical reactor, we must either adjust the size (B_g^2) or the core composition (B_m^2) such that $B_m^2 = B_g^2$

we also note,

$$B_m^2 > B_g^2 \quad \Rightarrow \quad \lambda_1 < 0 \quad \Rightarrow \quad \text{super critical}$$

$$B_m^2 = B_g^2 \quad \Rightarrow \quad \lambda_1 = 0 \quad \Rightarrow \quad \text{critical}$$

$$B_m^2 < B_g^2 \quad \Rightarrow \quad \lambda_1 > 0 \quad \Rightarrow \quad \text{sub critical}$$

$$B_g^2 = \left(\frac{\pi}{\tilde{a}}\right)^2$$

$$B_m^2 = \frac{v\Sigma_f - \Sigma_a}{D}$$

$$t \rightarrow \infty \quad \phi(x, t) \rightarrow A_1 \exp(-\lambda_1 t) \cdot \cos B_g x$$