Neutron Transport Theory Lecture Note (4)
-One-Speed diffusion Theory-

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4. The one-speed diffusion equation

4.1 Derivation of the diffusion equation

We will characterize the neutron distribution in the reactor by the neutron density $N(\mathbf{r},t)$ (or by the neutron flux $\phi(\mathbf{r},t)$)

We consider on arbitrary volume V of surface area S located anywhere within the reactor.

The total number of neutron in V at time t

$$\int_V d^3 r N(\mathbf{r},t) = \int_V d^3 r \frac{1}{v}\phi(\mathbf{r},t) \quad \cdots (1)$$

The time rate of change of the number of neutron in V

$$\frac{d}{dt}\left[\int_V d^3 r \frac{1}{v}\phi(\mathbf{r},t)\right] = \int_V d^3 r \frac{1}{v}\frac{\partial \phi}{\partial t}$$

$$= \text{Production in V} - \text{Absorption in V} - \text{Net leakage from V} \quad \cdots (2)$$

$$\text{Production in V} = \int_V d^3 r S(\mathbf{r},t) \quad \cdots (3)$$

$S(\mathbf{r},t)$: neutron source density

$$\text{Asorption in V} = \int_V d^3 r \Sigma_a(\mathbf{r})\phi(\mathbf{r},t) \quad \cdots (4)$$

$$\text{Net leakage from V} = \int_S d\mathbf{S}\cdot\mathbf{J}(\mathbf{r},t) \quad \cdots (5)$$

Convert the surface integral into volume integral by using Gauss's theory

$$\int_S d\mathbf{S}\cdot\mathbf{J}(\mathbf{r},t) = \int_V d^3 r \nabla\cdot\mathbf{J}(\mathbf{r},t) \quad \cdots (6)$$

$$\nabla \cdot \mathbf{J}(\mathbf{r},\mathrm{t}) = \mathrm{div}\mathbf{J}(\mathbf{r},\mathrm{t})$$

Substituting each of these mathematical expressions into Eq.(2),

$$\int_{V} \mathrm{d}^3\mathrm{r}\left[\frac{1}{v}\frac{\partial\phi}{\partial \mathrm{t}} - \mathrm{S} + \Sigma_{\mathrm{a}}\phi + \nabla\cdot\mathbf{J}\right] = 0 \qquad \cdots(7)$$

Eq.(7) must be hold for any volume V

$$\frac{1}{v}\frac{\partial\phi}{\partial \mathrm{t}} - \mathrm{S} + \Sigma_{\mathrm{a}}\phi + \nabla\cdot\mathbf{J} = 0 \qquad \cdots(8)$$

The equation contains two unknowns $\phi(\mathbf{r},\mathrm{t})$ and $\mathbf{J}(\mathbf{r},\mathrm{t})$. There is no exact relationship between $\phi(\mathbf{r},\mathrm{t})$ and $\mathbf{J}(\mathbf{r},\mathrm{t})$.

Diffusion approximation

$$\mathbf{J}(\mathbf{r},\mathrm{t}) \cong -\mathrm{D}(\mathbf{r})\nabla\phi(\mathbf{r},\mathrm{t}) \qquad \cdots(9)$$

where

$$\mathrm{D} = \frac{1}{3\Sigma_{\mathrm{tr}}} = \frac{1}{3\left(\Sigma_{\mathrm{t}} - \bar{\mu}_0\Sigma_{\mathrm{s}}\right)} \qquad \cdots(10)$$

(Diffusion coefficient)

$\bar{\mu}_0$: The average cosine of the scattering angle in a neutron scattering collision

Substituting Eq.(9) into Eq.(8),

$$\frac{1}{v}\frac{\partial\phi}{\partial \mathrm{t}} = \nabla\cdot\mathrm{D}(\mathbf{r})\nabla\phi(\mathbf{r}) - \Sigma_{\mathrm{a}}(\mathbf{r})\phi(\mathbf{r},\mathrm{t}) + \mathrm{S}(\mathbf{r},\mathrm{t}) \qquad \cdots(11)$$

(One-speed neutron diffusion equation)

If the D and Σ_{a} do not depend on position (homogeneous)

$$\frac{1}{v}\frac{\partial\phi}{\partial \mathrm{t}} - \mathrm{D}\nabla^2\phi + \Sigma_{\mathrm{a}}\phi(\mathbf{r},\mathrm{t}) = \mathrm{S}(\mathbf{r},\mathrm{t}) \qquad \cdots(12)$$

If the flux is not function of time (steady state)

$$-\mathrm{D}\nabla^2\phi + \Sigma_{\mathrm{a}}\phi(\mathbf{r}) = \mathrm{S}(\mathbf{r})$$

$$\cdots(13)$$

Dividing by $-\mathrm{D}$

$$\nabla^2\phi(\mathbf{r}) - \frac{1}{L^2}\phi(\mathbf{r}) = -\frac{S(\mathbf{r})}{D} \qquad \cdots(14)$$

where

$$L \equiv \sqrt{\frac{D}{\Sigma_a}} \qquad \text{(Diffusion length)}$$

L : A measure of how far the neutrons will diffuse from a source before they are absorbed.

〔Important〕

- The conditions that the diffusion approximation can be valid

① It is used to describe the neutron flux several mean free path away from the boundaries or isolated source

② The medium is only weakly absorbing

③ The neutron current is changing slowly on a time scale comparable to the mean time between neutron-nuclei collision

4.2 Initial and Boundary Conditions

(1)Initial condition

Specifying the neutron flux $\phi(\mathbf{r},0)$ for all positions $\mathbf{r}$ at the initial time t=0

$$\text{Initial condition}: \phi(\mathbf{r},0) = \phi_0(\mathbf{r})$$

(2)Boundary conditions

(a)Vacuum boundary

Ex. Outside boundary of a reactor

No neutrons can enter the reactor through this surface from outside.

A boundary condition that give correct neutron flux deep within the reactor where diffusion theory is valid is :

$$\phi(\tilde{x}_s) = 0$$

$$\tilde{x}_s = x_s + z_0 \qquad \text{(extrapolated boundary)}$$

where

$$z_0 = 0.7104\lambda_{tr} \qquad \text{(extrapolation length)}$$

$$\lambda_{tr} = \frac{1}{\Sigma_{tr}}$$

(b)Boundary condition for interfaces（material discontinuity）

① Continuity of the neutron flux $\phi(\mathbf{r}, t)$

② Continuity of the normal component of the neutron current density $\mathbf{J}(\mathbf{r},t)$

(c)The condition which should be satisfied always

$$0 \leq \phi(\mathbf{r}, t) < \infty \quad \text{(except in the neighborhood of localized sources)}$$

4.3 Neutron diffusion in nonmultiplying media

Example: Neutron flux $\phi(x)$ in an infinite homogeneous medium with plane source at the origin

Diffusion equation

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0、\quad x > 0$$

Boundary conditions

$$\text{(a)} \lim_{x\to 0^+} -D\frac{d\phi}{dx} = \frac{S_0}{2} \qquad S_0 : \text{neutron source}[s^{-1} \cdot cm^{-2}]$$

$$\text{(b)} \quad \lim_{x\to \infty} \phi(x) < \infty$$

The general solution

$$\phi(x) = A\exp\left(-\frac{x}{L}\right) + B\exp\left(\frac{x}{L}\right)$$

From boundary condition (b),

$$B = 0$$

From boundary condition (a),

$$\lim_{x\to 0^+} -D\left(-\frac{A}{L}\exp\left(-\frac{x}{L}\right)\right) = \frac{AD}{L} = \frac{S_0}{2}$$

$$\therefore \ A = \frac{S_0 L}{2D}$$

Hence $\quad \phi(x) = \dfrac{S_0 L}{2D}\exp\left(-\dfrac{x}{L}\right), \quad x > 0$

By symmetry

$$\phi(x) = \frac{S_0 L}{2D} \exp\left(\frac{x}{L}\right), \quad x < 0$$