Monotone Comparative Statics and Supermodular Games (June 29, July 3)

I. Comparative Statics

- Examples – important when making policy decisions
  - What happens to demand as income rises?
  - How does a firm's output level respond to an increase in the output price?
- Relationship between [optimal solution] and [parameters] $\rightarrow$ comparative statics
- Monotone comparative statics: when the optimal solution either monotonically increases or decreases as the parameters increase.

II. Objectives and Goals

- Consider the following problem

$$\max_{x \in X} f(x, \theta)$$

  where x is the decision variable, X is the feasible set, and θ is a parameter taken from a set Θ.
- $x^*(\theta)$: the solution to the maximization problem for a given θ
- Question: Under what conditions on f is $x^*(\theta)$ a nondecreasing function of θ?
- Use this theory to derive a new class of games in which a Nash equilibrium exists and the equilibria can be ordered.

III. Scalar Case (When $X \subset \mathbb{R}$ and $\Theta \subset \mathbb{R}$)

- Consider once again the objective:

$$\max_{x \in X} f(x, \theta)$$

  for each θ. Suppose further that X is compact and $f(\cdot, \theta)$ is a continuous function of x, so that the above maximization problem has a solution.
- For each θ, let $\arg\max f(x, \theta)$ be the set of maximizers.
- Need a concept for $\arg\max f(x, \theta)$ to be increasing as θ increases.

- A function $f : X \times \Theta \to \mathbb{R}$ satisfies **increasing differences** in $(x, \theta) \in X \times \Theta$ if for all $\theta', \theta \in \Theta$ such that $\theta' > \theta$ and for all $x, x' \in X$ such that $x' > x$,

$$f(x', \theta') - f(x, \theta') \geq f(x', \theta) - f(x, \theta).$$

Lemma. Suppose that $f : X \times \Theta \to \mathbb{R}$ is twice continuously differentiable. Then, f satisfies increasing differences in (x, θ) if and only if

$$\frac{\partial^2 f}{\partial x \partial \theta}(x, \theta) \geq 0.$$

- Let $\bar{x}(\theta)$ be the maximum element, and $\underline{x}(\theta)$ be the minimum element of $\arg\max f(x, \theta)$.

Theorem 1. Suppose that $f : X \times \Theta \to \mathbb{R}$ satisfies increasing differences. Then, $\bar{x}$ and $\underline{x}$ are nondecreasing functions of θ. That is, $\theta < \theta'$ implies $\bar{x}(\theta) \leq \bar{x}(\theta')$ and $\underline{x}(\theta) \leq \underline{x}(\theta')$.

- The above theorem holds when the constraint set X also depends on θ in the following way: $X(\theta) = [g(\theta), h(\theta)]$ where g and h are nondecreasing in θ and $g(\theta) \leq h(\theta)$.

- Stronger results can be obtained when f satisfies **strictly increasing differences.** A function $f : X \times \Theta \to \mathbb{R}$ satisfies **strictly increasing differences** in (x, θ) if for all $\theta, \theta' \in \Theta$ with $\theta' > \theta$ and for all $x, x' \in X$ with $x' > x$,

$$f(x', \theta') - f(x, \theta') > f(x', \theta) - f(x, \theta).$$

Theorem 1'. Suppose that $f : X \times \Theta \to \mathbb{R}$ satisfies strictly increasing differences. Then, every selection $x(\theta)$ from $\arg\max f(x, \theta)$ is a nondecreasing function.

- Another variant is when f satisfies decreasing differences. A function $f : X \times \Theta \to \mathbb{R}$ has **decreasing differences** if or all $\theta', \theta \in \Theta$ such that $\theta' > \theta$ and for all $x, x' \in X$ such that $x' > x$,

$$f(x', \theta') - f(x, \theta') \leq f(x', \theta) - f(x, \theta).$$

Theorem 1". Suppose that $f : X \times \Theta \to \mathbb{R}$ satisfies decreasing differences. Then, $\bar{x}$ and $\underline{x}$ are nonincreasing functions of θ.

IV. Examples

- Consumer Theory
  - Let there be two goods, whose consumption levels are denoted by x_1 and x_2. The utility function of the consumer is denoted by $U(x_1, x_2)$.
  - Notation: $U_1 \equiv \frac{\partial U}{\partial x_1}$, $U_2 \equiv \frac{\partial U}{\partial x_2}$
  - Let p_1 be price of good 1, p_2 be price of good 2, and m be income.
  - Conditions on whether each good is a normal good (its demand is nondecreasing in m.)
- Monopoly
  - A monopolistic firm faces a market demand function given by $D(p)$.
  - Let c be a (constant) per-unit cost of production.
  - Then, when a firm chooses a price p from the set $[c, \infty)$, profits are given by

$$\Pi(p, c) = (p - c)D(p)$$

V. Supermodular Games

- A game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is a **supermodular game** if
  - $S_i \subset \mathbb{R}$ is nonempty and compact for each $i \in N$
  - For each $s_{-i} \in S_{-i}$, $u_i(\cdot, s_{-i})$ is continuous in its own argument $s_i \in S_i$.
  - u_i satisfies increasing differences in (s_i, s_j) for all $j \neq i$ (in which case it is assumed that all other s_k, $k \neq i, k \neq j$ are fixed and u_i is seen as a function of (s_i, s_j)).
- As before let β_i be the best-response correspondence of $i \in N$. Consider two selections – the largest best response $\bar{b}$ and the smallest best response $\underline{b}$. That is,

$$\bar{b}(s_{-i}) = \max \beta_i(s_{-i}),\ \underline{b}(s_{-i}) = \min \beta_i(s_{-i}).$$

- From Theorem 1, $\bar{b}(\cdot)$ and $\underline{b}(\cdot)$ are nondecreasing functions of s_{-i}.

VI. Tarski's Fixed Point Theorem and Nash Equilibria of Supermodular Games

- Just as Kakutani's fixed point theorem was useful, the following fixed point theorem is useful.

Tarski's Fixed Point Theorem: Suppose

- X a nonempty compact interval of $\mathbb{R}^n$.
- $f : X \to X$ nondecreasing. That is, $x \leq y \Rightarrow f(x) \leq f(y)$.

Then, there exists $x \in X$ such that $f(x) = x$. Moreover, there exists a smallest and largest fixed point.

- The fixed point theorem does not hold when f is nonincreasing instead of nondecreasing.

Theorem 2. A supermodular game admits at least one Nash equilibrium. Moreover, there exists a largest and smallest Nash equilibrium. Formally, there exists $\bar{s} \in S$ and $\underline{s} \in S$ such that for every Nash equilibrium $s \in S$,

$$\underline{s}_i \leq s_i \leq \bar{s}_i, \ \forall i \in N$$

VII. Iterated Removal of Strictly Dominated Strategies

- It is also shown in Milgrom and Roberts (1990) (as a special case), within the set of strategies that survive the iterated removal of strictly dominated strategy is the smallest and largest Nash equilibria.
- The largest and smallest equilibria also play a key role in the outcome of the set of strategies that survive the iterated removal of strictly dominated strategies.

Theorem 3. Let G be a supermodular game, and let S_i^* denote the set of strategies for each $i \in N$ which survive the iterated removal of strictly dominated strategies (Version 1). Then for each $i \in N$ and $s_i \in S_i^*$,

$$\underline{s}_i \leq s_i \leq \bar{s}_i.$$

Moreover, if the game G has a unique Nash equilibrium, then it is also dominance solvable.

- While the result is presented here for iterated removal of strictly dominated strategies, the original result includes a broader class of adjustment processes (which the authors called adaptive dynamics.)

VIII. Examples of Supermodular Games

- Arms Race (outlined in Milgrom and Roberts (1990))
  - $N = \{1, 2\}$ and each country chooses a level s_i of arms.
  - Payoff given by
  $$u_i(s_i, s_j) = -C(s_i) + B(s_i - s_j).$$
  where $C(s_i)$ is a smooth function, and $B(s_i, s_{-i})$ is a smooth concave function
- Cournot Duopoly
- Bertrand Duopoly

IX. Other Topics and References

- Most of notes from Amir (2005), which is a survey on some of the results on complementarity and economics.
- Results for n-dimensional space is given in the following appendix. Results also hold for more abstract partially ordered sets.
- Results on lattice theory and consumer theory (Mirman and Ruble (2008), Antoniadou (2007))
- Summary of results on supermodular games (Vives (1990))
- Tarski's fixed point theorem for correspondences and alternative proof of the existence of Nash equilibria and the structure of the set of Nash equilibria – (Zhou (1994))
- Procedure to find all Nash equilibria of supermodular games (Echenique (2007))
- Supermodular mechanism design (Mathevet (2010))

- Quasi-supermodularity (ordinal concept of supermodularity) and single-crossing (ordinal concept of increasing differences) (Milgrom and Shannon (1994))
  - An increasing transformation of a supermodular function need not be supermodular.

Appendix A: The Case of n-dimensional Space ($\mathbb{R}^n$) – Definitions

- Let $x, y \in \mathbb{R}^n$, where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$. Define the ordering $\leq$ by $x \leq y \Leftrightarrow x_i \leq y_i \ \forall i = 1, 2, \cdots, n$.
- In contrast to the case when $n = 1$, when $n \geq 2$, not all x and y are comparable. For example, $n = 2$, $x = (0, 1)$ and $y = (1, 0)$, neither $x \leq y$ nor $y \leq x$ holds.
- Given $x, y \in \mathbb{R}^n$, define the operation $\vee$ by

$$x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \cdots, \max\{x_n, y_n\})$$

- In words, $x \vee y$ is an element in $\mathbb{R}^n$ where each ith component is given by either x_i or y_i, whichever is larger. $x \vee y$ is called the **join** of x and y.
- Note that $x \leq x \vee y$ and $y \leq x \vee y$, and for all z with $x \leq z$ and $y \leq z$, $x \vee y \leq z$. (In such a case, $x \vee y$ is called the **supremum** of $\{x, y\}$.)
- Define the operation $\wedge$ by

$$x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \cdots, \min\{x_n, y_n\})$$

- $x \wedge y$ is called the **meet** of x and y.
- Note that $x \wedge y \leq x$ and $x \wedge y \leq y$, and for all z with $z \leq x$ and $z \leq y$, $z \leq x \wedge y$. (In such a case, $x \wedge y$ is called the **infimum** of $\{x, y\}$.)
- A subset $L \subset \mathbb{R}^n$ is a **sublattice** if for every $x, y \in L$, $x \wedge y$ and $x \vee y$ are both in L.
- Examples
  1. $L_1 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ is a sublattice.
  2. $L_2 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\}$ is not a sublattice. Both $(1, 0)$ and $(0, 1)$ are in L but $(1, 1) = (1, 0) \vee (0, 1)$ is not.

3. $L_3 = \{(x_1, x_2) \in \mathbb{R}^2 | 0 \leq x_1 \leq x_2 \leq 1\}$ is a sublattice.
4. $L_4 = \{(x_1, x_2) | 0 \leq x_1 \leq 1/2, 0 \leq x_2 \leq 1/2\} \cup \{(x_1, x_2) | 1/2 \leq x_1 \leq 1, 1/2 \leq x_2 \leq 1\}$ is a sublattice.

- Suppose that X is a sublattice of $\mathbb{R}^n$. A function $f : X \to \mathbb{R}$ is **supermodular** if for every $x, y \in X$,
$$f(x) + f(y) \leq f(x \vee y) + f(x \wedge y)$$

Proposition 1. *Let $f : X \to \mathbb{R}$ be a supermodular function. The set of maximizers of f, $X^* = \{y \in X | f(y) = \max_{x \in X} f(x)\}$ is a sublattice of $\mathbb{R}^n$.*

Corollary 1. *Let $f : X \to \mathbb{R}$ be a continuous supermodular function and suppose that X is a compact sublattice of $\mathbb{R}^n$. Then, there exist a largest maximizer $\bar{x}$ and a smallest maximizer $\underline{x}$. That is, if $x \in X^* = \{y \in X | f(y) = \max_{x \in X} f(x)\}$, then $\underline{x} \leq x \leq \bar{x}$.*

- Caution: This does not mean that all maximizers can be ordered.

B. Increasing Differences and Monotonicity of Maximizers for the General Case

- Let Θ denote the set of parameters and X denote the set of decision variables. Then,

Proposition 2. *Suppose that $f : X \times \Theta \to \mathbb{R}$ is a supermodular function on (x, θ). Then, f satisfies increasing differences in (x, θ).*

- It can be shown that if f satisfies increasing differences in $(x, \theta) \in X \times \Theta$, then it satisfies increasing differences in $(\theta, x) \in \Theta \times X$. Although mathematically equivalent, the order matter when establishing which set consists of decision variable, and which set consists of parameters.

- Increasing differences is related to monotone comparative statics – the set of solutions is "increases" as θ increases.

- Let $\Phi : \Theta \to X$ be a correspondence. Φ is said to be **increasing** if for every θ and θ' in Θ such that $\theta' \geq \theta$, $x \in \Phi(\theta)$ and $x' \in \Phi(\theta')$ imply

$$x \wedge x' \in \Phi(\theta) \ \text{ and } \ x \vee x' \in \Phi(\theta').$$

- For a function f, define $X^*(\theta) = \arg\max_{x \in X} f(x, \theta)$ to be the set of maximizers of f.

Proposition 3. *Suppose that $f(\cdot, \theta)$ is supermodular in X for each $\theta \in \Theta$, and let $f : X \times \Theta \to \mathbb{R}$ satisfy increasing differences. Then, the correspondence that assigns to each $\theta \in \Theta$ the set of maximizers $X^*(\theta)$ is increasing.*

Corollary 2. *Suppose that $f(\cdot, \theta)$ is supermodular in X for each $\theta \in \Theta$, and let $f : X \times \Theta \to \mathbb{R}$ satisfy increasing differences and suppose that for each θ, $X^*(\theta)$ has a smallest and largest element. Let $\underline{x} : \Theta \to X$ and $\bar{x} : \Theta \to X$ be such that $\underline{x}(\theta)$ is the smallest element in $X^*(\theta)$ and $\bar{x}(\theta)$ is the largest element in $X^*(\theta)$. Then, the functions $\underline{x}$ and $\bar{x}$ are nondecreasing functions of θ. That is,*

$$\theta \leq \theta' \Rightarrow \underline{x}(\theta) \leq \underline{x}(\theta') \text{ and } \bar{x}(\theta) \leq \bar{x}(\theta').$$

- Definition of increasing differences extended to $\mathbb{R}^n$

- f satisfies **increasing differences** if for any distinct i, j and two vectors $x = (x_i, x_j, x_{-ij})$ and $x' = (x'_i, x'_j, x_{-ij})$ with $x'_j \geq x_j$ and $x'_i \geq x_i$,

$$f(x'_i, x'_j, x_{-ij}) - f(x_i, x'_j, x_{-ij}) \geq f(x'_i, x_j, x_{-ij}) - f(x_i, x_j, x_{-ij})$$

  where $x_{-ij} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$. That is, f satisfies increasing differences iff it satisfies the first definition for any two x_i and x_j.

- The analogue of Proposition 2 is the following.

Proposition 4. *Suppose that $f : X \to \mathbb{R}$ is a supermodular function on (x, θ). Then, f satisfies increasing differences.*

- Moreover, in this setting where $X \subseteq \mathbb{R}^n$, supermodularity and increasing differences are equivalent.

Proposition 5. *Suppose that $f : X \to \mathbb{R}$ satisfies increasing differences on $X \subseteq \mathbb{R}^n$. Then, f is supermodular on X.*

- Characterization of supermodularity and increasing differences in terms of derivatives is given below.

Proposition 6. *Let f be a twice continuously differentiable function. Then, f satisfies increasing differences if and only if for all $i \neq j$,*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$$

- The above definition gives a simple way to check increasing differences (and supermodularity from Proposition 5) when the objective function is twice continuously differentiable.

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