Proof of the Existence Theorem (June 22)

I. Review

- Definition of Nash equilibrium
- Mixed extension
- This lecture – proof of the existence theorem using Kakutani's fixed point theorem.

II. Limits, Compact Sets and Convex Sets

- For $x, y \in \mathbb{R}^m$, define the **distance** between x and y by

$$d(x,y) = \left(\sum_{i=1}^{m} (x_i - y_i)^2 \right)^{1/2}$$

- Let $\{x^k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^m$. The sequence $\{x^k\}_{k=1}^{\infty}$ is said to **converge to** x (denoted by $x^k \to x$) if for every $\epsilon > 0$, there exists a number N such that for all $k \geq N$, $d(x^k, x) < \epsilon$. In this case, x is said to be the **limit** of the sequence $\{x^k\}_{k=1}^{\infty}$ and is denoted by $x = \lim_{k\to\infty} x^k$.
- Useful properties of limits:

Suppose $\{x^k\}_{k=1}^{\infty}$ and $\{y^k\}_{k=1}^{\infty}$ are two sequences in $\mathbb{R}^m$ such that $x^k \to x$ and $y^k \to y$ where $m \in \mathbb{N}$.

1. If $\{z^k\}_{k=1}^{\infty}$ is a sequence such that $z^k = x^k + y^k$ for all k, then $z^k \to (x + y)$.
2. Suppose $m = 1$ and consider the sequence $\{z^k\}_{k=1}^{\infty}$ where $z^k = x^k \cdot y^k$. Then, $z^k \to x \cdot y$.
3. If $x^k \geq y^k$ for all k, then $x \geq y$.
4. For each x^k, let $x^k = (x_1^k, x_2^k, \cdots, x_m^k)$ be written out component-wise, and let $x = (x_1, x_2, \cdots, x_m)$. Then, $x^k \to x$ if and only if $x_i^k \to x_i$ for each $i = 1, 2, \cdots, m$.

- A set $X \subset \mathbb{R}^m$ is **closed** $\Leftrightarrow$ for every sequence $\{x^k\}_{k=1}^{\infty} \subset X$ such that $x^k \to x$, then $x \in X$.

- A set $X \subset \mathbb{R}^m$ is **bounded** $\Leftrightarrow$ there exists M such that $|x_i| \leq M$ for every $x = (x_1, x_2, \cdots, x_n) \in X$ and $i = 1, 2, \cdots, n$

- A set $X \subset \mathbb{R}^m$ is **compact** $\Leftrightarrow$ X is both closed and bounded.

- Equivalently, a set $X \subset \mathbb{R}^m$ is **compact** if for every sequence $\{x^k\}_{k=1}^{\infty}$ such that $x^k \in X$ for all k, there exists a subsequence $\{x_{k(q)}\}_{q=1}^{\infty}$ such that $x_{k(q)} \to x \in X$.

- A set $X \subset \mathbb{R}^m$ is **convex** $\Leftrightarrow$ for every $x, x' \in X$ and $\lambda \in [0, 1]$, $(1 - \lambda)x + \lambda x' \in X$.

III. Continuity – Functions and Correspondences (Set-valued Functions)

- Let $f : X \to Y$ be a function where $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^l$. f is said to be **continuous** if for every sequence $\{x^k\}_{k=1}^{\infty} \subset X$, $x^k \to x \Rightarrow f(x^k) \to f(x)$. That is, the sequence $\{f(x^k)\}_{k=1}^{\infty}$ converges to $f(x)$.

- A famous result involving continuous functions and compact sets.

Weierstrauss' Theroem: Let $f : X \to \mathbb{R}$ be a continuous function where X is a (nonempty) compact subset of $\mathbb{R}^m$. Then, there exist $\underline{x}, \bar{x} \in X$ such that

$$f(\underline{x}) \leq f(x) \leq f(\bar{x}) \; \forall x \in X.$$

- $\Phi : X \to Y$ is a **correspondence** if for every $x \in X$, $\Phi(x)$ is a subset of Y. That is, $\Phi(x) \subset Y$. (β_i, the best-response correspondence, is one such example.)

- Suppose that X and Y are compact. Φ is **upper hemicontinuous** (or in some texts, upper semicontinuous) if $\Phi(x)$ is compact for all $x \in X$ and for every sequence $\{x^k\}_{k=1}^{\infty} \subset X$ such that $x^k \to x$ and every sequence $\{y^k\}_{k=1}^{\infty} \subset Y$ such that $y^k \in \Phi(x^k)$ for each k and $y^k \to y \Rightarrow y \in \Phi(x)$.

- Upper hemicontinuity is one extension of the concept of continuity to correspondences.

IV. Kakutani's Fixed Point Theorem and the Existence Theorem

Kakutani's Fixed Point Theorem Suppose that

- X is a compact, convex, and <u>nonempty</u> subset of $\mathbb{R}^m$, and

- the correspondence $\Phi : X \to X$ is upper hemicontinuous
- for each $x \in X$, $\Phi(x)$ is a nonempty, compact, and convex subset of $\mathbb{R}^m$

Then, there exists $x \in X$ such that

$$x \in \Phi(x).$$

- A direct application of Kakutani's Fixed Point Theorem $\Rightarrow$ Existence of a Nash Equilibrium in mixed strategies. Specifically,
  - $X := \prod_{i \in N} \Delta(S_i)$ is a compact and convex set and also nonempty.
  - In place of Φ, define $\beta(\sigma) = \beta_1(\sigma_{-1}) \times \beta_2(\sigma_{-2}) \times \cdots \times \beta_n(\sigma_{-n})$. Then, $\beta : \prod_{i \in N} \Delta(S_i) \to \prod_{i \in N} \Delta(S_i)$ as a correspondence is
    1. nonempty-valued, compact-valued, and convex-valued (for each σ, $\beta(\sigma)$ is a nonempty, compact, and convex set)
    2. upper hemicontinuous.
- Similar techniques can be used to prove the existence of other equilibria.

Reference Notes:

- Original papers: Nash (1950) (Proof based on Kakutani's fixed point theorem) and Nash (1951) (Proof based on Brouwer's fixed point theorem)
- Mas-Colell, Whinston, and Green (1995) (Chapter 8, Section 8.D and Appendix A)
- Vega-Redondo (2003) (Chapter 2, pp. 35–45)

References

Mas-Colell, A., M. Whinston, and J. Green (1995). *Microeconomic Theory*. New York: Oxford University Press.

Nash, J. F. (1950). Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences of the United States of America 36*, 48–49.

Nash, J. F. (1951). Non-cooperative games. *Annals of Mathematics, Second Series 54*, 286–295.

Vega-Redondo, F. (2003). *Economics and the Theory of Games*. Cambridge University Press.