Iterated Removal of Dominated Strategies (June 15, June 19)

I. Review

- Game in strategic form: $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ where
  - N: set of players
  - S_i: set of strategies of player $i \in N$
  - u_i: payoff function of player $i \in N$
- Mixed extension of G: $(N, (\Delta(S_i))_{i \in N}, (\pi_i)_{i \in N})$ where
  - $\Delta(S_i)$: set of mixed strategies (probability distributions over S_i)
  - π_i: expected payoff function of player $i \in N$

II. Strictly Dominated Strategies and Weakly Dominated Strategies

Definition. Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic form game. A strategy of player i, $s_i \in S_i$ is said to be **strictly dominated by** $s'_i \in S_i$ if for all $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) < u_i(s'_i, s_{-i})$$

A strategy s_i is said to be **strictly dominated** if it is strictly dominated by some $s'_i \in S_i$.

- Notation:
  - $s_{-i} := (s_1, s_2, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n)$ (strategy combination where the ith component is taken out)
  - $S_{-i} := \prod_{j \neq i} S_j = S_1 \times S_2 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$
- Interpretation: s_i is always a worse strategy compared to s'_i in terms of the payoffs that can be realized based on the choice of strategies of the other players.
- Assumption of rationality – rational players do not choose strategies that are strictly dominated. From this point forward, assume that players are rational in that way.
- Example: Prisoner's Dilemma (reproduced below)

$A \setminus B$	C	D
C	$-2, -2$	$-6, 0$
D	$0, -6$	$-5, -5$

C is strictly dominated by D for both players.

- If both players are rational, then it is expected that these players both choose D and not C. However, both players can do better by both choosing $C \rightarrow$ "dilemma."
- In the other examples of this handout – no player has a strictly dominated strategy.
- A strategy $s_i \in S_i$ for player $i \in N$ is said to be **strictly dominant** if it strictly dominates all other strategies $s_i' \in S_i$ $(s_i' \neq s_i)$.
- A weaker form of domination – weak domination (definition given below).

Definition. Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic form game. A strategy of player i, $s_i \in S_i$ is said to be **weakly dominated by** $s_i' \in S_i$ if for all $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) \leq u_i(s_i', s_{-i})$$

and for some $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})$$

A strategy s_i is said to be **weakly dominated** if it is weakly dominated by some $s_i' \in S_i$. A strategy $s_i \in S_i$ for player $i \in N$ is said to be **weakly dominant** if it weakly dominates all other strategies $s_i' \in S_i$ $(s_i' \neq s_i)$.

Consider the game below.

$A \setminus B$	L	R
U	$1, 1$	$1, 0$
D	$1, 0$	$0, 0$

- For player 1, D is weakly dominated by U but not strictly dominated.
- For player 2, R is weakly dominated by L but not strictly dominated.

III. Definitions for Mixed Extensions and Equivalent Results

- Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game where S_i is a finite set for all $i \in N$ and consider its mixed extension $G' = (N, (\Delta(S_i))_{i \in N}, (\pi_i)_{i \in N})$.

- A mixed strategy σ_i is **strictly dominated** by another mixed strategy σ_i' in the mixed extension G' if

$$\pi_i(\sigma_i, \sigma_{-i}) < \pi_i(\sigma_i', \sigma_{-i}), \ \forall \sigma_{-i} \in \prod_{j \neq i} \Delta(S_j).$$

- The following is a simpler equivalent form.

Proposition 1. A mixed strategy σ_i is strictly dominated by another mixed strategy σ_i' if and only if

$$\pi_i(\sigma_i, s_{-i}) < \pi_i(\sigma_i', s_{-i}), \ \forall s_{-i} \in S_{-i}.$$

- The next statement is a word of caution.

Caution. Let $s_i \in S_i$ be a strategy that is not strictly dominated in the game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$. Then, s_i as a mixed strategy can still be strictly dominated by some $\sigma_i' \in \Delta(S_i)$ in the mixed extension G'.

- However, the following does hold.

Proposition 2. Suppose that $s_i \in S_i$ is strictly dominated in the game G. Then any mixed strategy σ_i such that $\sigma_i(s_i) > 0$ is strictly dominated in the mixed extension G'.

IV. Iterated Removal of Strictly Dominated Strategies – Example

$1 \setminus 2$	L	C	R
U	$3, 3$	$2, 1$	$0, 0$
M	$2, 2$	$2, 1$	$0, 0$
D	$0, 1$	$0, 1$	$0, 0$

- Suppose that both players are rational in that they do not choose strictly dominated strategies. Moreover, suppose that each player knows that the other player is rational.

- In the above example, none of the strategies for player 1 (U, M, D) are strictly dominated. So, supposing that player 1 is rational, player 1 still may choose U, M, or D.

- Strategy R of player 2 is strictly dominated by L and C. L nor C is strictly dominated. Therefore, player 2, if rational, will not choose R.

$1 \setminus 2$	L	C	R
U	$3,3$	$2,1$	$0,0$
M	$2,2$	$2,1$	$0,0$
D	$0,1$	$0,1$	$0,0$

- Suppose that player 1 knows that player 2 is rational. Then, player 1 knows that player 2 will not choose R. Now, because both players know that player 2 will not choose R, the game is reduced to the following:

$1 \setminus 2$	L	C
U	$3,3$	$2,1$
M	$2,2$	$2,1$
D	$0,1$	$0,1$

- In the reduced game, D is now strictly dominated by M. So, if player 1 knows that player 2 is rational, player 1 will not choose D. If player 2 also knows that player 1 knows that player 2 is rational, then both players know that player 1 will not choose D, and the game is reduced to the following:

$1 \setminus 2$	L	C
U	$3,3$	$2,1$
M	$2,2$	$2,1$

- In the game above, strategy C is strictly dominated by L. Therefore, if player 2 knows that player 1 knows that player 2 is rational, then player 2 will not choose C. If player 1 also knows that player 2 knows that player 1 knows that player 2 is rational, then in both players' minds, the game is reduced to the following with now C deleted:

$1 \setminus 2$	L
U	$3,3$
M	$2,2$

- Now, M is strictly dominated by U. Therefore, if player 1 knows that player 2 knows that player 1 knows that player 2 is rational, then player 1 will not choose M. If player 2 also knows that player 1 knows that player 2 knows that player 1 knows that player 2 is rational, then both players can deduce that the strategy combination (U, L) results:

$1 \setminus 2$	L
U	$3, 3$

- The process described above → iterative removal of strictly dominated strategies
- Because the knowledge of rationality assumed for this process is complex, it is convenient instead to assume the following.

Common knowledge of rationality: Assume any chain (including infinite ones) of "Player 1 knows that player 2 knows that ⋯ (infinitely long)."

V. Iterated Removal of Strictly Dominated Strategies – General Procedure

- Suppose throughout this section that the set of strategies S_i for each $i \in N$ is finite.

Version 1: Delete All Strictly Dominated Strategies

1. Step 1: For all $i \in N$, delete all such $s_i \in S_i$ that are strictly dominated. Let S_i^1 denote the set of strategies that remain.
2. Step 2: Consider now the game with S_i^1 as the set of strategies for each $i \in N$. Delete all such $s_i \in S_i^1$ that are strictly dominated by some $s_i' \in S_i^1$. Let S_i^2 denote the set of strategies that remain.
3. Continue the process until there are no strategies that are strictly dominated.

- If G is a game such that the above process stops and yields a unique strategy combination $(s_1^*, s_2^*, \cdots, s_n^*) \in \prod_{i \in N} S_i$, then the game G is said to be **dominance solvable**.
- The following proposition is useful in showing an important property of this process for finite games.

Proposition 3. Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game in strategic form. For each $i \in N$, let $A_i \subseteq S_i$ and consider the game $G^T = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ where u_i is the same as in G, restricted to the set $\prod_{i \in N} A_i$. Let $s_i \in A_i$ be strictly dominated by some $s'_i \in A_i$ in the game G. Then, s_i is also strictly dominated by s'_i in the game G^T.

- This result implies that it does not matter whether **all** strictly dominated strategies or just **one** strictly dominated strategy is deleted in one step.
- It also does not matter whether **all** players delete their strictly dominated strategies in one step or just **one** player deletes his/her strictly dominated strategies in one step.
- From above, we can define two alternative versions, both leading to the same set of strategies in the end.

Version 2: Delete Only One Strictly Dominated Strategy

1. Step 1: Choose **one** player $i \in N$ who has a strictly dominated strategy. Delete **one** $s_i \in S_i$ that is strictly dominated. Let S_i^1 denote the set of strategies that remain, and for the remaining players $j \neq i$, let $S_j^1 = S_j$.
2. Step 2: Consider now the game with S_i^1 as the set of strategies for each $i \in N$, choose **one** $i \in N$ and delete **one** $s_i \in S_i^1$ that is strictly dominated by some $s'_i \in S_i^1$. Let S_i^2 denote the set of strategies that remain for player i and let $S_j^2 = S_j^1$ for all other players $j \neq i$.
3. Continue the process until no player has a strictly dominated strategy.

VI. Iterated Removal of Weakly Dominated Strategies

- Review of definition of weak domination:

A strategy of player i, s_i is said to be **weakly dominated** by another strategy s'_i if for all $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) \leq u_i(s'_i, s_{-i})$$

and for some s_{-i}, the above inequality holds with a strict inequality $<$. That is, for some $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})$$

- If s_i is weakly dominated by s_i', then choosing s_i is **never better** and in at least one case **worse** than choosing s_i'.
- By replacing "strictly" with "weakly" in each version, one can think of anaologues of the two versions for strict domination.

Version 1W: Delete All Weakly Dominated Strategies

1. Step 1: For all $i \in N$, delete all such $s_i \in S_i$ that are weakly dominated. Let S_i^1 denote the set of strategies that remain.
2. Step 2: Consider now the game with S_i^1 as the set of strategies for each $i \in N$. Delete all such $s_i \in S_i^1$ that are weakly dominated by some $s_i' \in S_i^1$. Let S_i^2 denote the set of strategies that remain.
3. Continue the process until there are no strategies that are weakly dominated.

Version 2W: Delete Only One Weakly Dominated Strategy

1. Step 1: Choose one $i \in N$, and do the following. Delete only one $s_i \in S_i$ that is weakly dominated. Let S_i^1 denote the set of strategies that remain.
2. Step 2: Choose another $i \in N$, do the following. Considering now the game with S_i^1 as the set of strategies for each $i \in N$, delete $s_i \in S_i^1$ that are weakly dominated by some $s_i' \in S_i^1$. Let S_i^2 denote the set of strategies that remain.
3. Continue the process until there are no strategies that are weakly dominated.

- The strategies that remain after version 1W and 2W may not be the same, even if each player has a finite number of strategies – that is, even if S_i is a finite set for all players.

- The order in which the players are chosen in version 2W also affects which strategies remain in the end.
- Main reason – Proposition 3 fails to hold if "strictly dominated" replaced by "weakly dominated."

VII. Never Best Response and Rationalizability

- A closely related concept to strict domination is concept of a strategy being never a best response.

Definition. Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game in strategic form. A strategy $s_i \in S_i$ for player i is said to be a **best response** to $s_{-i} \in S_{-i}$ if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \ \forall s_i' \in S_i.$$

A strategy $s_i \in S_i$ is **never a best response** if there does not exist $s_{-i} \in S_{-i}$ to which s_i is a best response.

- If a strategy s_i is strictly dominated, then it is never a best response.
- Consider now iterated removal of strategies that are never best responses.

Iterated Removal of Strategies that are Never Best Responses:

1. Step 1: For all $i \in N$, delete all such $s_i \in S_i$ that are never best responses. Let S_i^1 denote the set of strategies that remain.
2. Step 2: Consider now the game with S_i^1 as the set of strategies for each $i \in N$. Delete all such $s_i \in S_i^1$ that are never best responses. Let S_i^2 denote the set of strategies that remain.
3. Continue the process until there is not a strategy that is never a best response.

- The end result is a set of strategies that are said to be **rationalizable**. See Bernheim (1984) and Pearce (1984).
- For two-player games (mixed extension of a finite game in strategic form), set of rationalizable strategies coincides with the set of strategies that survive the iterated removal of strictly dominates strategies. (Never best response = strictly dominated)

- For more than two players, the equivalence result may not hold. However, a strategy that is strictly dominated must never be a best response.

References

Bernheim, B. D. (1984). Rationalizable strategic behavior. *Econometrica 52*(4), 1007–1028.

Pearce, D. G. (1984). Rationalizable strategic behavior and the problem of perfection. *Econometrica 52*(4), 1029–1050.