Games of Incomplete Information and Bayesian Games (July 24, July 27)

I. Games of Complete Information and Games of Incomplete Information

- Up until this point, it was assumed that all players knew completely the structure of the game, and that each player knew that the other players knew this as well $\cdots$ $\rightarrow$ **complete information game** (not to be confused with perfect information for extensive form games)

- What if some parts of the game – actions, payoffs, etc. – was not known to some of the players?

- In such a case, the game is said to be of **incomplete information**. Tools introduced so far cannot be directly applied to such situations

- Today's topic: Bayesian games. Convert a game of incomplete information $\rightarrow$ a game of complete information (with imperfect information) called a **Bayesian game**.

II. A Conversion into Bayesian Games

- Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be the base game (similar to the stage game for repeated games).

- Consider the case in which the payoffs may not be known. For each $i \in N$, introduce the set of types T_i. The set of types are used to define the different possibilities of outcomes. Assume that for each i, T_i is a finite set.

- The game starts with an artificial player, named "Nature" selects the types of all the players $t = (t_1, t_2, \cdots, t_n) \in \prod_{i \in N} T_i \equiv T$. Nature has no payoff function so that there is no incentive behind choosing a particular type.

- Nature selects the types randomly so that it is *a*s if Nature is playing a behavioral strategy by specifying a probability distribution over the set of types. Formally, Nature chooses $(t_1, t_2, \cdots, t_n)$ with probability $p(t_1, t_2, \cdots, t_n)$ where p is the (joint) distribution function. p is often called the **prior distribution**. This p is known to all players.

- Formally, the prior distribution is a function $p : T \rightarrow \mathbb{R}$ such that $p(t) \geq 0$ and $\sum_{t \in T} p(t) = 1$.

- After the types are chosen, each player can observe his/her own type but not of the other players.
- A rough sketch of the game tree will be drawn in class.
- Recall that a strategy is defined as a function on the set of information sets to the set of actions. A strategy in a Bayesian game, denoted by s_i, is then a function from T_i to A_i. That is, it associates to each type $t_i \in T_i$ an action in A_i, denoted by $s_i(t_i)$. Let S_i be the set of strategies for each $i \in N$ and let $S \equiv \prod_{i \in N} S_i$.
- The payoff function u_i of the base game now depends on the types of all players. $u_i : A \times T \to \mathbb{R}$ where $A \equiv \prod_{i \in N} A_i$.
- The payoff function of the Bayesian game is given by the expectation of the payoffs over all types and is defined as a function over the set of strategy combinations. The assumption is that these payoffs are calculated at the beginning of the game – before Nature chooses a combination of types.
- Formally, the expected payoff function of i, denoted by $Eu_i : S \to \mathbb{R}$ is given by the following equation.

$$Eu_i(s_1, s_2, \cdots, s_n) = \sum_{t \in T} p(t_1, t_2, \cdots, t_n)\, u_i\left((s_1(t_1), s_2(t_2), \cdots, s_n(t_n)), (t_1, t_2, \cdots, t_n)\right)$$

III. Bayesian Nash Equilibrium – Two Definitions

Definition 1. A strategy profile s^* is said to be a **Bayesian Nash equilibrium (BNE)** if it a Nash equilibrium of the Bayesian game. That is, for all $i \in N$ and for all $s_i \in S_i$,

$$Eu_i(s_i^*, s_{-i}^*) \geq Eu_i(s_i, s_{-i}^*)$$

- The way in which the expected utility is taken suggests that players themselves do not know of their own types when these values are computed. This is because at the beginning of the game, Nature has not selected the types of all the players. Thus, the expectation for each player is taken as if they do not know of their types.
- However, in the incomplete information setting, each player knows his/her own type but not of the others.

- Below is a definition of Bayesian Nash equilibrium for that scenario.

Definition 1'. A strategy profile s^* is said to be a **Bayesian Nash equilibrium (BNE)** if for all $i \in N$, for all $t_i \in T_i$, and for all $a_i \in A_i$,

$$Eu_i(s_i^*(t_i), s_{-i}^*|t_i) \geq Eu_i(a_i, s_{-i}^*|t_i)$$

where Eu_i represents player i's conditional expected payoff, given that player i's type is t_i.

- The expectation is taken over the conditional probability distribution of the other players' types t_{-i} given that the player i's type is t_i.
- Review of joint distribution, marginal distribution, conditional probability distribution
  - Let $t \in T$ and recall $p(t)$ where $t = (t_1, t_2, \cdots, t_n)$ represents the probability that player 1's type is t_1, player 2's type is t_2, $\cdots$, player n's type is t_n. It is assumed that
$$p(t) \geq 0 \ \forall t \in T \quad \text{and} \quad \sum_{t \in T} p(t) = 1.$$
    p is called the joint probability distribution of $(t_1, t_2, \cdots, t_n)$
  - Given p, the marginal distribution of t_i, denoted by p_i, denotes the probability that player i's type is t_i. That is,
$$p_i(t_i) = \sum_{t'_{-i} \in T_{-i}} p(t_i, t'_{-i})$$
  - The conditional probability of the other players' types given that i's type is t_i, denoted by $p(t_{-i}|t_i)$ is given by the following formula:
$$p(t_{-i}|t_i) = \frac{p(t_i, t_{-i})}{p_i(t_i)} = \frac{p(t_i, t_{-i})}{\sum_{t'_{-i} \in T_{-i}} p(t_i, t'_{-i})}$$
  - When the types are independent, then the joint distribution p satisfies the following:
$$p(t_1, t_2, \cdots, t_n) = p_1(t_1)p_2(t_2) \cdots p_n(t_n)$$
- It can be shown that these two definitions are equivalent.

- All the above can be extended to the case when there are a continuum of types of players.
  - Instead of p, Nature selects the types of the players from a joint (cumulative) distribution function F, usually assumed to have a density function f.
  - Summation ($\sum$) should be replaced by integration ($\int$)

IV. Example

- 2 players $N = \{1, 2\}$. $A_1 = \{C, D\}$ and $A_2 = \{C, D\}$. Player 2 is informed which game is played, but player 1 only knows that prisoner's dilemma is played with probability 1/3, and chicken game is played with probability 2/3. The two games are given below (the same numbers as in the first lecture).

$1 \setminus 2$	C	D
C	$-2, -2$	$-6, 0$
D	$0, -6$	$-5, -5$

$1 \setminus 2$	C	D
C	$0, 0$	$-2, 2$
D	$2, -2$	$-5, -5$

- To formulate this situation into a Bayesian game, introduce the following types. Player 1 only has one type $T_1 = \{t_1\}$. Player 2 has two types, corresponding to the two different games: $T_2 = \{t_2, t_2'\}$ where t_2 corresponds to the prisoner's dilemma, and t_2' corresponds to the chicken game.

- $u_1((C, C), (t_1, t_2)) = -2$, while $u_1((C, C), (t_1, t_2')) = 0$ etc. (Some more examples may be computed in class.)

- Strategy for player 1: Choose C or D; strategy for player 2: Choose C or D in the prionser's dilemma, choose C or D in the chicken game. Denote by $(x - y)$ as the strategy in which player 2 chooses $x \in \{C, D\}$ in the prisoner's dilemma and $y \in \{C, D\}$ in the chicken game.

- To compute BNE from Definition 1, need to compute the Eu_i for $i = 1, 2$. This is summarized below.

$1 \setminus 2$	$C - C$	$D - C$	$C - D$	$D - D$
C	$-2/3, -2/3$	$-2, 0$	$\underline{-2}, 2/3$	$\underline{-10/3}, \underline{4/3}$
D	$\underline{4/3}, -10/3$	$\underline{-1/3}, \underline{-3}$	$-10/3, -16/3$	$-5, -5$

- Two BNE: $(D, D - C)$ and $(C, D - D)$

- Check by Definition 1' that these two are BNE under that definition.

V. Continuum of Types – First-price Auction

- Consider a first-price sealed-bid auction.
  - Each bidder submits a bid in a sealed envelope so that no other bidder can see that bid.
  - The bidder with the highest bid wins the object and pays the amount that he/she bid. The other bidders do not win the object but does not have to pay.
  - Let v_i be bidder i's valuation and b_i be the bid made by i. If i wins the object, i's payoff is $v_i - b_i$; if i does not win the object, i's payoff is 0.
- Suppose there are n bidders. Each bidder knows his/her own valuation v_i but not of the other players.
- These valuations correspond to the types in a Bayesian game. Assume that each bidder's type is drawn randomly from a uniform distribution with support $[0, 1]$ ($F(v) = v$) and independently. This implies that the highest valuation of a bidder is 1, while the lowest is 0.
- Strategy of i: a function $\beta_i : [0, 1] \to \mathbb{R}_+$ such that $\beta_i(v_i)$ represents the bid made by bidder i when his/her valuation is v_i.
- Objective: to find a Bayesian Nash equilibrium of this game $(\beta_1^*, \beta_2^*, \cdots, \beta_n^*)$ such that
  - it is symmetric – that is each bidder uses the same strategy ($\beta_1^* = \beta_2^* = \cdots = \beta_n^*$)
  - it is differentiable with respect to v_i
  - it is strictly increasing in v_i
- It can be shown that there exists such a Bayesian Nash equilibrium and $\beta_1^* = \beta_2^* = \cdots = \beta_n^* = \beta^*$ where β^* is given by the following.

$$\beta^*(v) = \frac{n-1}{n} v$$

VI. Signaling

- $N = \{1, 2\}$: the set of players. Consider player 1 as a "worker" and player 2 as a "firm."

- Suppose that player 1 has two possible types: $T_1 = \{H, L\}$, where H denotes "high ability," and L denotes "low ability." Player 2 has only one type – description of player 2's types will be omitted in this section. Player 2 does not know the type of player 1. Suppose $p(H) = p(L) = 0.5$.
- The firm (player 2) has to set wages for the worker and has two choices: W (high wage) or w (low wage). Suppose that regardless of type, player 1's payoff when receiving a high wage is 6, while a low wage is 3.
- The payoffs to the firm are given by the following table:

	High	Low
W	6	1
w	2	5

- Because player 2 does not know whether player 1 is a high-ability work or low-ability worker but does know the prior distribution, player 2 can calculate the expected payoff from offering W or w:
  - W: $(1/2) \times 6 + (1/2) \times 1 = 7/2$
  - w: $(1/2) \times 5 + (1/2) \times 3 = 8/2 = 4$

  Therefore, player 2 should offer w.
- This is inefficient, and the high-ability type would prefer to be able to show that he/she has high ability $\rightarrow$ signal
- As an example, suppose that each type has the option of going to school (S) or not (N)
- The cost of schooling: 2 for high ability, 5 for low ability.
- Game tree will be drawn in class.
- Two sequential equilibria:
  1. $(S - N, W - w, ((1, 0), (0, 1)))$ (separating equilibrium)
  2. $(S - S, w - w, ((p, 1 - p), (1/2, 1/2)))$ where $p \leq 1/2$ (pooling equilibrium)
- First equilibrium is such that schooling acts as an effective signal such that by observing it, player 2 now can figure out which type player 1 is from this information.

- Second equilibrium is such that schooling is not an effective signal.
- This second equilibrium is not "intuitive," especially the belief $p \leq 1/2$. Suppose that player 2 observes that player 1 had received schooling. A low ability player 1 cannot benefit from doing so as the possible payoffs of 1 and -2 are below 6 and 3, so this player must be a high-ability player, and the belief with $p \leq 1/2$ does not seem to make sense (although, it is consistent).
- Refinement of sequential equilibrium related to the previous argument for signaling games: Cho and Kreps (1987)

VII. Other Topics

- Corresponding sections in the text:
  - Bayesian games: Mas-Colell, Whinston, and Green (1995), Section 8.E
- Sufficient condition for the existence of a pure strategy Bayesian Nash equilibrium
  - Earlier papers: Radner and Rosenthal (1982), Milgrom and Weber (1985)
  - Techniques using supermodularity: Vives (1990), Athey (2001)
- Robust equilibrium
  - a refinement of Nash equilibrium such that it is robust to changes in the information of the players, defined in Kajii and Morris (1997)
  - the maximizer of exact potential is a robust equilibrium (Ui (2001))
- Global games (Carlsson and van Damme (1993))
  - Global games involve incomplete information in payoff of the game, but each player receives a (private) signal regarding that unknown payoff. Based on that signal, each player then formulates beliefs about the other players' signals and what type of actions they might take.
  - Common example:

$1 \setminus 2$	Invest	Not Invest
Invest	θ, θ	$\theta - 1, 0$
Not Invest	$0, \theta - 1$	$0, 0$

  - The value of θ is not known to either player, but the prior distribution is known.

- Each player $i \in \{1, 2\}$ independently receives some signal s_i regarding the value of θ.
- applications in monetary economics – Morris and Shin (1998), Rochet and Vives (2004)

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