Finitely and Infinitely Repeated Games (July 17)

I. Prisoner's Dilemma Revisited

- Consder once again the prisoner's dilemma. For today's lecture, the following payoff matrix is used.

$1 \setminus 2$	C	D
C	$6, 6$	$0, 8$
D	$8, 0$	$2, 2$

- Choosing D is rational, but usually in everyday life, people do not always betray others.

- Reason: The assumption that the game is played only once may be a key factor.

- Question: How does the analysis change when the same game is played more than once? $\rightarrow$ repeated games.

- Keywords: Finitely repeated games, Infinitely repeated games, Folk Theorem

II. Prisoner's Dilemma Repeated Twice

- Suppose now that the prisoner's dilemma game is repeated twice. (A game tree will be drawn in class)

- The payoff for each player i in the repeated game is given by the following formula.

$$\text{(Payoff in the first game)} + \delta_i \text{ (Payoff in the second game)}$$

δ_i: **discount factor** of player i, $0 < \delta_i < 1$. Payoff in the second game is worth δ_i times the payoff in the first game.

- Another interpretation of δ_i: exogenously determined probability that the repeated game continues. (At each repetition, the game then ends with probability $1 - \delta_i$.)

- A strategy for a player is given in the form
  - Choose from $\{C, D\}$ to play in the first prisoner's dilemma game.
  - For *each* outcome $((C, C), (C, D), (D, C), (D, D))$, assign C or D. Essentially, the second part of the strategy is a function from $\{C, D\} \times \{C, D\}$ to $\{C, D\}$.

– $2 \cdot 2^4 = 2^5 = 32$ pure strategies for each player.

- The following fact can be checked directly.

In the twice-repeated prisoner's dilemma, the following strategy taken by each player is the unique subgame-perfect equilibrium:

- Play D first.
- Play D in the second game, regardless of what happens in the first game.

Informally, this strategy is described by the phrase, "always play D."

III. Finitely Repeated Games

- Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be the **component game** or **stage game**, which is to be repeated. It is assumed that within a repetition, each player i chooses an action in S_i independently and simultaneously.
- At the end of each repetition, each player is informed of the choices made by the other players.
- Concept of a strategy:
  - For the twice repeated prisoner's dilemma game – the decision node for player 1, when the game is played a second time, can be characterized by the sequence of actions taken previously = history. Moreover, player 2's information set in the second repetition can also be characterized by the same history.
  - History at time t, h^t, is such that $h^t = (a^1, a^2, \cdots, a^{t-1})$ where for each $1 \leq \tau \leq t-1$, $a^\tau \in S := \prod_{i\in N} S_i$. $h^1 := \emptyset$ denotes the "empty" history. Let H^t be the set of possible histories at the beginning of the t-th repetition. Here we assume that this set is the same for all players since at the beginning of each repetition, the information given to them is the same.
  - A strategy for each $i \in N$ is given by the sequence $\sigma_i = (\sigma_i^1, \sigma_i^2, \cdots)$ such that for each $t \geq 2$, $\sigma_i^t : H^t \to S_i$. For $t = 1$, for notational ease, let $\sigma_i^1 \in S_i$ for all $i \in N$.
- Payoffs –
  - Payoff function defined on terminal node.

- Each terminal node can be characterized by a sequence of action profiles → Payoff defined on sequence of action profiles.
- Payoffs are discounted so that payoffs for player i in the t-th repetition is worth δ_i^{t-1} as much as payoffs in the 1st repetition, where $0 < \delta_i < 1$.
- Let $(a^1, a^2, \cdots, a^T)$ be a sequence of action profiles. Then, player i's payoff function U_i in the T-repetitions of the game G is given by

$$U_i(a^1, a^2, \cdots, a^T) = \sum_{t=1}^{T} \delta_i^{t-1} u_i(a^t)$$

- The previous result for the prisoner's dilemma can be generalized in the following way.

Proposition 1. *Let G be a strategic form game with a unique Nash equilibrium a^*. Then, the game G repeated T times, where T is a positive integer, has a unique subgame-perfect equilibrium in which each player i plays a_i^* each time, regardless of what happens in the previous repetitions of G.*

- In terms of the prisoner's dilemma, as long as it is repeated a finite number of times, the only subgame-perfect equilibrium is for each player to always choose D.
- Same result as in the original game
- Key assumptions:
  - Unique Nash equilibrium in the stage game. (See Question 1 for when there is more than one Nash equilibrium.)
  - Finite repetition – different set of results for infinitely repeated games.

IV. Infinitely Repeated Games – Formal Definition

- Infinitely repeated game – approximation of a situation in which the end of the game is not known for certain.
- The concept of strategy is defined in the same manner as the previous section.
- Need a new set of definitions because the length of the game is no longer finite – no terminal nodes.

- Payoffs now defined on an infinite sequence of action profiles.
  - Let $(a^1, a^2, \cdots)$ be an infinite sequence of action profiles. Then, the payoffs in the infinitely repeated game is given by

$$U_i(a^1, a^2, \cdots) = \sum_{t=1}^{\infty} \delta_i^{t-1} u_i(a^t)$$

  - Because payoffs of an infinite stream may not seem intuitive, average payoffs are used and is defined by

$$(1 - \delta_i) \sum_{t=1}^{\infty} \delta_i^{t-1} u_i(a^t)$$

- There are many subgame-perfect equilibria of an infinitely repeated game G, even if G has only one Nash equilibrium $\rightarrow$ Folk Theorem (to be discussed in the next section).
- To check subgame-perfect equilibrium – "one-shot deviation principle" (in supplementary material).

V. Folk Theorem

- Once again, consider the prisoner's dilemma as an example. Consider the following strategy, which is often called a **trigger** strategy:
  - Choose C first.
  - If at least one player chooses D at some t^*, then choose D for t-th repetition, where $t \geq t^* + 1$.
- The "trigger" refers to when someone (including the player himself/herself) chooses D, from which point the player chooses D forever.
- Because one such deviation from C initiates an unending punishment of taking D, the strategy above is sometimes called the "grim trigger" strategy.

Fact 1. For δ_i sufficiently close to 1 for all $i \in N$, the trigger strategy defined above played by both players is a Nash equilibrium in the infinitely repeated prisoner's dilemma game.

- The above result is a special case of the "folk theorem." To state the folk theorem, define for each $i \in N$ the minimax value of the game G:

$$v_i = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

- Below is a "classical" folk theorem

Theorem 1. *Let G be a stage game, and let v_i be the minimax value of player $i \in N$. Take any outcome $a \in \prod_{i \in N} S_i$ such that $u_i(a) > v_i$ for all $i \in N$. Then, there exists $\bar{\delta}$ with $0 < \bar{\delta} < 1$ such that for all $\delta_i > \bar{\delta}$, there exists a Nash equilibrium σ^* such that the average payoff for player $i \in N$ equals $u_i(a)$.*

- Note above theorem stated in terms of Nash equilibrium $\rightarrow$ How about subgame-perfect equilibrium?

Fact 2. For δ_i sufficiently close to 1 for all $i \in N$, the trigger strategy defined above is a subgame-perfect equilibrium in the infinitely repeated prisoner's dilemma game.

- The above result is a special case of the following.

Proposition 2. *Let G be a stage game and let a^* be a Nash equilibrium of G. Let $a \in \prod_{i \in N} S_i$ be an action profile such that $u_i(a) > u_i(a^*)$ for all $i \in N$. Then, there exists $\bar{\delta} \in (0, 1)$ such that for all $\delta_i > \bar{\delta}$, there exists a subgame-perfect equilibrium with average payoff equal to $u_i(a)$ for each $i \in N$.*

- General results:
  - Extend the results of Theorem 1 to any payoffs in the convex hull of possible payoffs.
  - Under the additional condition of full dimensionality of the above convex hull or in the case of two players, Theorem 1 can be strengthened by replacing "Nash equilibrium" with "subgame-perfect equilibrium."

- The strategy used – a trigger strategy with two punishment stages.

VI. Other Topics and Literature

- Folk theorems listed here can be found in Fudenberg and Maskin (1986). Proposition 2 is from Friedman (1971).
- Optimal penal code and characterization of subgame-perfect equilibrium
  - In Abreu (1988), a simplified concept of strategy is used to analyze subgame-perfect equilibrium – (simple) penal code.
  - The strategy prescribes what is to be taken initially, what is to be taken if someone defects – strategy described through paths of actions.
  - This technique is used to calculate the subgame-perfect equilibria of the infinitely repeated Cournot duopoly games in Abreu (1986).
- Renegotiation-proof equilibrium (Farrell and Maskin (1989), Bernheim and Ray (1989))
  - Players are allowed at each stage to renegotiate on the strategy being played.
  - For example, once the trigger is implemented, would it be in the interest for all players involved to follow the grim trigger strategy with this renegotiation option.
- Imperfect monitoring
  - To implement the trigger strategy, deviations need to be detected. In the folk theorem, it is assumed that each player can detect a deviation once it occurs.
  - A weakening of this assumption is that such monitoring is not perfect – deviations may not be detected.
  - Each player receives a noisy signal about the actions of the others.
  - Folk theorem in this case (Fudenberg, Levine, and Maskin (1994)), using sequential equilibria (the topic of next lecture).
- Payoffs: $\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} u_i(a^t)$ instead of discounting. (The folk theorem in this case called "The Aumann-Shapley/Rubinstein folk theorem" in Fudenberg and Maskin (1986).)

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