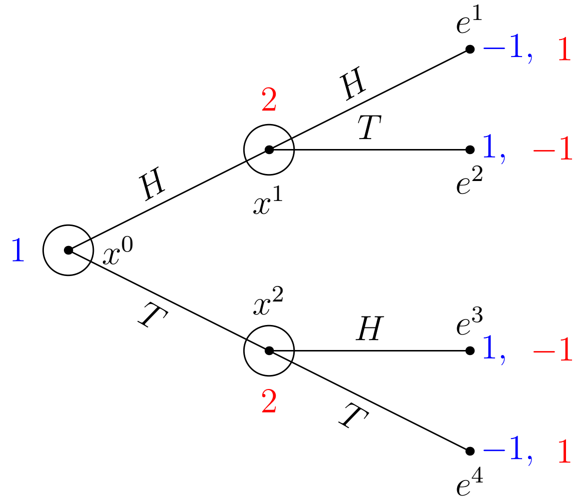
Extensive Form Games and Subgame-perfect Equilibrium (July 10, July 13)

I. Extensive Form Games – Example

- Consider the matching coins example with the following modification.

  1. Player 1 chooses H or T first.
  2. Player 2 chooses H or T next after seeing whether player 1 has chosen H or T.

- Below is a game tree of the example:



- How to read this diagram

  – Progression of play: $\longrightarrow$ (from left to right)

  – First, player 1 chooses H or T

  – After player 1 has chosen H or T, player 2 chooses H or T

  – The numbers at the right represent the payoffs for each player when that node is reached. For example, if player 1 chooses H and player 2 chooses T, player 1 receives a payoff of 1, and player 2 receives a payoff of -1.

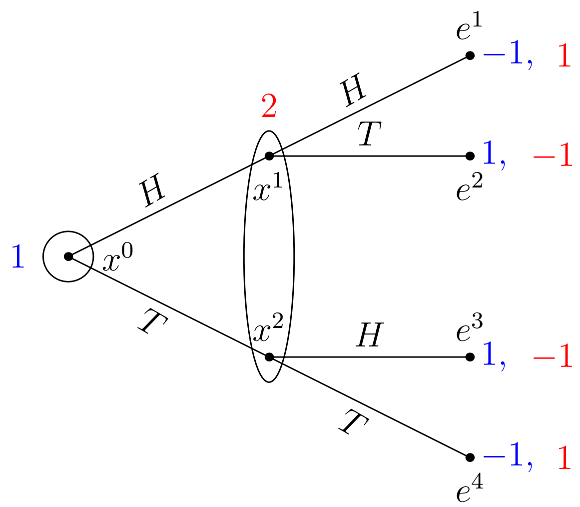
  – Convention: In the payoff combination, the first number is player 1's payoffs, the second number is player 2's payoffs, etc.

- Consider the matching coins example with the following modification.

  1. Player 1 chooses H or T first.

2. Player 2 chooses H or T next but **does not know** whether player 1 has chosen H or T.

- The game tree of the modified game is given in the following.



II. Observations from the Example

- Progression of play depicted by a tree. In graph-theoretic terms, a tree has the following characteristics.
  - There is a root, called the initial node, where the play of the game begins.
  - There are nodes at the end of the game to which payoffs of both players are assigned.
  - A tree is a connected graph – every node is connected by an edge – with no loops.
  - For each node, there exists a unique path (to be formally defined later) that starts from the root.
- Decision nodes – nodes where a player chooses which action to take
- Information sets – the "oval" around the decision nodes where if there are several nodes in the same information set, the player who chooses an action at those nodes cannot tell which node he/she is.
- A game is called a **game of perfect information** if each information set is a singleton. Otherwise, the game is of **imperfect information**.

  - The first version of matching coins is a game of perfect information.
  - The "modified" version is a game of imperfection information.

III. Formal Definition of a Game in Extensive Form

- Like strategic form games, extensive form games are defined by several components.
- The set of players $N = \{1, 2, \cdots, n\}$.
- The set of **decision nodes** X and **terminal nodes** E.
  - There exists a transitive ordering $\prec$ on $X \cup E$ where for $x, x' \in X$, $x \prec x'$ represents that node x **precedes** x' (and equivalently, x' **succeeds** x). Also, define the ordering $\preceq$ on X where $x \preceq x'$ represents that $x \prec x'$ or $x = x'$.
  - There exists $x^0 \in X$ such that for all $x \in X \cup E$, $x^0 \preceq x$. This node x^0 is called the **initial node**.
  - The nodes in E are precisely those $e \in E$ such that there is no x such that $e \prec x$.
  - A node x is said to **immediately precede** another node x' if there does not exist $x'' \in X$ such that $x \prec x'' \prec x'$. Also, x is said to be the **immediate predecessor** of x'. Assume that there is only one such immediate predecessor for each node x.
- For each $x \in X$, let $\iota(x) = i$ denote the player $i \in N$ who moves at decision node x.
- For each $x \in X$, denote by $A(x)$ the **set of actions** that are available at decision node x. Let $A := \bigcup_{x \in X} A(x)$ denote the set of actions that are available throughout the game.
- For each $x \in (X \setminus \{x^0\}) \cup E$, define $\alpha(x)$ to be the action that leads to the node x. $\alpha(x) \in A(x')$ where x' is the immediate predecessor of x. In terms of graphs, $\alpha(x)$ denotes the edge that leads into x. Also, α satisfies the property that if $x' \neq x''$ are two nodes that immediately follow some decision node x, then $\alpha(x') \neq \alpha(x'')$.
- Let $\mathcal{I}$ be a partition[1] of X such that for each $i \in N$, if x and x' are in the same set in the partition, then
$$\iota(x) = \iota(x') \text{ and } A(x) = A(x')$$

[1] A **partition** of X is a set $\{H_1, H_2, \cdots, H_K\}$ such that (a) for all $k \neq k'$, $H_k \cap H_{k'} = \emptyset$ and (b) $\bigcup_{k=1}^{K} H_k = X$.

Each set in the partition is called an **information set**.

- The **payoff function** $u_i : E \to \mathbb{R}$ for each $i \in N$.

A game in **extensive form** or **an extensive form game** G is defined as the combination of the above components – that is, $G = (N, X \cup E, \preceq, \iota, \mathcal{I}, (A(x))_{x \in X}, \alpha(x)_{x \in (X \setminus \{x^0\}) \cup E} (u_i)_{i \in N})$

- In the matching coins example:
  - $N = \{1, 2\}$
  - $X = \{x^0, x^1, x^2\}$ and $E = \{e^1, e^2, e^3, e^4\}$. Moreover,
    * $x^0 \prec x^1 \prec e^1$
    * $x^0 \prec x^1 \prec e^2$
    * $x^0 \prec x^2 \prec e^3$
    * $x^0 \prec x^2 \prec e^4$
  - $\iota(x^0) = 1$, $\iota(x^1) = \iota(x^2) = 2$.
  - $\mathcal{I} = \{\{x^0\}, \{x^1\}, \{x^2\}\}$.
  - $A(x^0) = A(x^1) = A(x^2) = \{H, T\}$
  - $\alpha(x^1) = H$, $\alpha(x^2) = T$, $\alpha(e^1) = H$, $\alpha(e^2) = T$, $\alpha(e^3) = H$, $\alpha(e^4) = T$.
  - Payoffs:
    * $u_1(e^1) = -1, u_1(e^2) = 1, u_1(e^3) = 1, u_1(e^4) = -1$.
    * $u_2(e^1) = 1, u_2(e^2) = -1, u_2(e^3) = -1, u_2(e^4) = 1$.

- Unless otherwise stated, assume that the sets N, X, E, and $A(x)$ for each $x \in X$ are finite sets, in which case we call G a **finite extensive form game**.

- Examples of games that are not finite:
  - Stackelberg duopoly: a Cournot duopoly model, in which one firm chooses an output level before the other firm and the other firm can observe the quantity choice of the first firm. (The game tree will be drawn on the board.)
  - Infinitely repeated games: Details will be discussed later. In this case, the set X is infinite, and E cannot have the interpretation given above.

- *Very rarely* are games in extensive form formally defined in this way in research papers nowadays.

IV. Formal Definition of a (Pure) Strategy

- Denote by $\mathcal{I}(x)$ the information set that contains node x.
- Let $\mathcal{I}_i := \{\mathcal{I}(x) | x \in X, \iota(x) = i\}$. That is, $\mathcal{I}_i$ represents the set of all information sets that include decision nodes at which player $i \in N$ moves.

> A **(pure) strategy** of a player $i \in N$ is a function $s_i : \mathcal{I}_i \to A$ such that for each $x \in X$, $s_i(\mathcal{I}(x)) \in A(x)$.

- For a game of perfect information, a strategy s_i is a function that assigns to each node x such that $\iota(x) = i$, an action in $A(x)$. That is, if $X_i = \{x \in X | \iota(x) = i\}$, then $s_i : X_i \to A$ with $s_i(x) \in A(x)$.
- In the matching coins example:
  - A strategy for player 1 consists of choosing either H or T at node x^1.
  - A strategy for player 2 consists of choosing either H or T at node x^2 **and** choosing either H or T at node x^3. An example of a strategy is $H - T$, which means that player 2 chooses H at x^2 and T at x^3.

V. Extensive Form → Strategic Form

- For each $i \in N$, let S_i be the set of strategies, as defined in Section IV.
- Once a combination of strategies $(s_1, s_2, \cdots, s_n)$ is chosen, a particular sequence of actions results that is induced by these strategies, leading to a terminal node t. Then, with abuse of notation, define $u_i(s_1, s_2, \cdots, s_n) = u_i(t)$. In such a way, the payoff function can be defined over the set of strategy combinations $S := \prod_{i \in N} S_i$.
- The combination of $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is the strategic form game associated with the extensive form game, defined in Section III.
- For each extensive form game, there is a unique strategic form game representation. The converse need not be true.

- In von Neumann and Morgenstern (1953), the strategic form is defined as a simplified form of the extensive form (although the formal components and terminology are different).

VI. Behavioral Strategy, Mixed Strategy, and Perfect Recall

- As in strategic form games, randomization can be introduced in the concept of strategy in several ways.
  - Probability distribution over strategies. $\rightarrow$ **mixed strategy**
  - In the definition of strategy – at each information set, instead of choosing an action with certainty, could randomize over the actions. Formally, for $i \in N$, consider a mapping b_i such that for $H \in \mathcal{I}_i$, $b_i(H) \in \Delta(A(x))$ for $x \in H$, where $\Delta(\cdot)$ denotes taking the probability distribution over the set in $(\cdot)$. $\rightarrow$ b_i is called a **behavioral strategy** for player i.
- In general, [mixed strategy] $\neq$ [behavioral strategy], but for each behavioral strategy, there exists a mixed strategy such that the probability that each terminal node is reached is the same for both strategies.
- When the game satisfies perfect recall (which is explained next), for each mixed strategy combination, there is a behavioral strategy such that the probability that each terminal node is reached is the same.
- A game is said to satsify perfect recall if no player forgets what he/she has known in an earlier part of the game – including the actions of others and of that player himself/herself that have been taken before. (The formal definition can be found in the footnote on page 224 of Mas-Colell, Whinston, and Green (1995).
- Assume that the games we consider from here on satisfy perfect recall. By the above observation, for a finite extensive game with perfect recall, there must exist a Nash equilibrium in behavioral strategies.

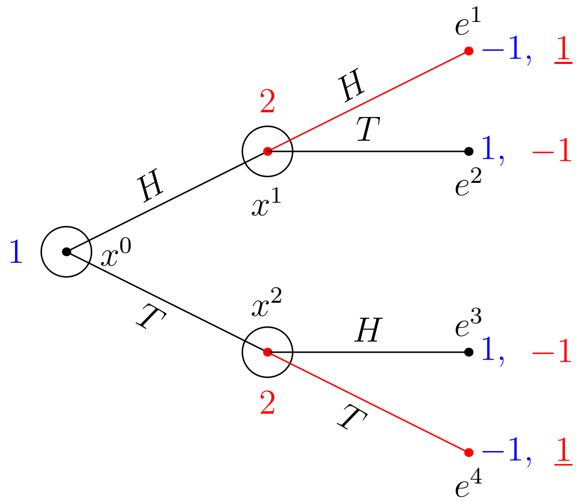
VII. Backwards Induction for Games of Perfect Information

- A game in extensive form is said to be a game of perfect information if each information set is a singleton. That is, for each $x \in X$, $\mathcal{I}(x) = \{x\}$.

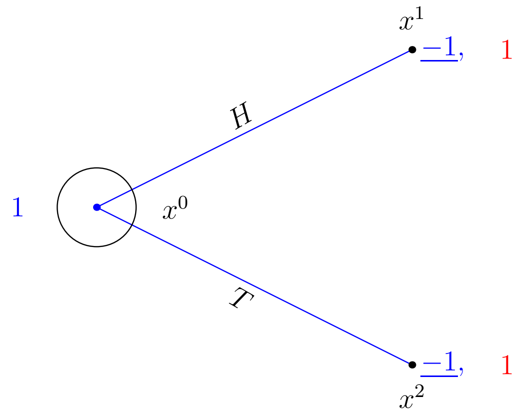
Backwards Induction:

- Start at a "penultimate" ("second to last") node x. Formally, take a decision node x such that there is no $x' \in X$ such that $x \prec x'$. Thus, all nodes that follow x must be terminal nodes in E.
- The player who moves at x, $\iota(x)$, chooses an action $A(x)$ that leads to a terminal node with the highest payoff for player $\iota(x)$. Assign that action to $s_{\iota(x)}(x)$. Let $u^x = (u_1^x, u_2^x, \cdots, u_n^x)$ denote the payoffs that result, where u_i^x denotes the payoff of player i in the payoff vector u^x. Assign the payoff $u^x \in \mathbb{R}^n$ to that node x, and make x a terminal node by deleting the rest of the game that follows x.
- Do the above process for all such penultimate nodes x.
- Repeat the process, now with the penultimate nodes of the shortened game. The process repeats until the initial node is reached.

- The illustration of the above process on the first matching coins game.
  - The penultimate nodes in the above game are nodes x^1 and x^2. At both x^1 and x^2, player 2 moves. At x^1, H gives a higher payoff to player 2 than T. Therefore, according to the backwards induction process, choose H at x^1. Similarly, at x^2, choose T. These choices are drawn in red in the diagram below.



– Now, replace the tree that follows x^1 by the payoff $(-1, 1)$, and what follows x^2 with the payoff $(-1, 1)$. These payoffs result from the choices that were outlined in red above. The resulting game tree is given below.



– In the reduced game above, player 1 has two choices, H and T, which lead to identical payoffs. Therefore, both actions are optimal, so that both edges are colored blue in the game tree above.

– Backwards induction yields the following strategy profiles: $(H, H - T)$ and $(T, H - T)$.

Fact 1. For a game of perfect information, the strategy combination that is obtained from backwards induction is a Nash equilibrium.

- Sketch of proof

– Suppose s^* is obtained from backwards induction, but not a Nash equilibrium. Then, there exists $i \in N$ such that s_i^* is not a best response to s_{-i}^*.

– This implies that at some $x \in X$ with $\iota(x) = i$, player i can obtain a higher payoff by choosing a different action than $s_i^*(x)$.

– Choose such a $x \in X$ such that no x' with $x \prec x'$ and $\iota(x') = i$ has the above properties.

– Consider the reduced game in which x is the penultimate node with the payoffs resulting from the choices in $A(x)$. Such can be appropriately defined since all the other players' strategies are given by s_{-i}^*, and player i's choices that follow must be the same as s_i^*.

– However, these payoffs need not be $u_i(s_i^*, s_{-i}^*)$ and $u_i(s_i', s_{-i}^*)$, but the following relation still holds:

$$\text{[Payoff from choosing a different action]} > \text{[Payoff from choosing } s_i^*(x)\text{]}$$

– This contradicts the fact that $s_i^*(x)$ is chosen by backwards induction.

VIII. Subgames and Subgame-perfect Equilibrium

A **subgame** G' starting from x is defined by the following components.

- The root x must be such that $\mathcal{I}(x) = \{x\}$. That is, the information set containing x is a singleton.

- G' contains all nodes that follow x. That is, for all x' with $x \prec x'$, x' is a node in G'.

- Let x' be a node such that $x \prec x'$. Then, for all $y \in \mathcal{I}(x')$, y is contained in G'.

Intuitively, a subgame by itself can satisfy the conditions for a game in extensive form.

- **Warning**: It is <u>not sufficient</u> to just consider subtrees – that is, a part of the original tree that is itself a tree.

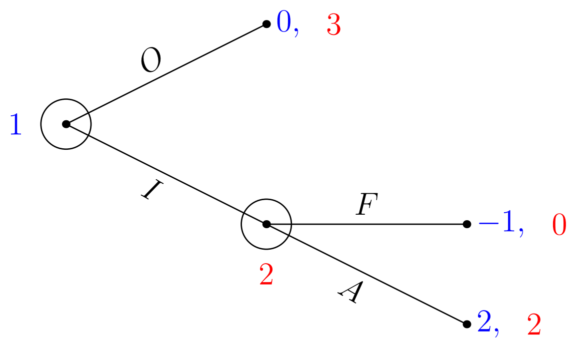
- By definition, a game is a subgame of itself. Thus, for any extensive form game, there exists at least one subgame. We call a subgame G' of a game G to be a **proper subgame** if G' is a subgame and $G' \neq G$.

- A strategy profile $s^* = (s_1^*, s_2^*, \cdots_n^*)$ is a **subgame-perfect equilibrium** (SPE) of game G for each subgame G' of G, the restriction of s^* to G' is a Nash equilibrium.

- From the above observation, every subgame-perfect equilibrium is by definition a Nash equilibrium.

- There are some strategy profiles that are Nash equilibria but not subgame-perfect equilibria. The classic example of such a case is "the chainstore paradox" and the concept of "incredible threats."

- The chainstore paradox consists of two players, each player being a firm. Firm 1 is an entrant, while firm 2 is the incumbent. The game tree is given below.



  - The actions I stands for "In," O for "Out," F for "Fight," and A for "Accomodate."
  - The strategic form of the game is given below.

$1 \setminus 2$	F	A
O	$\underline{0}, \underline{3}$	$0, \underline{3}$
I	$-1, 0$	$\underline{2}, \underline{2}$

  - There are two Nash equilibria: (I, A) and (O, F). (I, A) is a subgame-perfect equilibrium, while (O, F) is not. Therefore, player 2 choosing F is explained as an incredible threat. (O, F) as a Nash equilibrium makes sense that player 1 does not want to enter if it believes that player 2 would choose F. However, once player 2's decision node is actually reached, player 2 would not choose F, thus making the "threat" of choosing F seem incredible (or not believable).

- The next result states that the strategy combination from backwards induction yields a subgame-perfect equilibrium.

Fact 2. For a game of perfect information, the strategy combination that is obtained from backwards induction is a subgame-perfect equilibrium.

IX. Backwards Induction for Games of Imperfect Information

- In the previous sections, backwards induction was defined for games of perfect information. For games of imperfect information, it is not possible to look at one node sepearately from another node if they are in the same information set.
- A modified version of backwards induction is given below:

Generalized Backwards Induction:

- Take a decision node x such that the subgame starting from node x does not contain a proper subgame. Label such a subgame G_x.
- Find a Nash equilibrium of G_x, and let u^x denote the payoffs at that Nash equilibrium. Assign the payoff $u^x \in \mathbb{R}^n$ to that node x, and make x a terminal node by deleting the rest of the game that follows x.
- Do the above process for all x such that the subgame starting from x does not contain a proper subgame.
- Repeat the process, now with the shortened game. The process repeats until the initial node is reached.

- By a similar reasoning, the strategy profiles from the above process yields a subgame-perfect equilibrium.
- Because most examples are of imperfect information, generalized backwards induction is simply called backwards induction.
- Important example of games of imperfect information $\rightarrow$ repeated games

Fact 3. For a game of imperfect information, the strategy combination that is obtained from generalized backwards induction is a subgame-perfect equilibrium.

X. Reading Notes

- Different texts have different versions and notations when defining extensive form games:
  - Mas-Colell et al. (1995): Section 7.C, 9.B (pp. 268 – 279)
  - Vega-Redondo (2003): 1.1 – 1.5 (pp. 1 – 23), 4.1 – 4.3 (pp. 110 – 117)

References

Mas-Colell, A., M. Whinston, and J. Green (1995). *Microeconomic Theory*. New York: Oxford University Press.

Vega-Redondo, F. (2003). *Economics and the Theory of Games*. Cambridge University Press.

von Neumann, J. and O. Morgenstern (1953). *Theory of Games and Economic Behavior* (Third ed.). Princeton: Princeton University Press.