Potential Games (June 26, June 29)

I. Common Payoff Games

- Example

$1 \setminus 2$	L	R
U	$3, 3$	$0, 0$
D	$0, 0$	$1, 1$

- There are two Nash equilibria: (U, L) and (D, R)
- Easier to find Nash equilibria of these games – one of which is a strategy combination that yields the highest payoff to both players.
- Equilibrium can be found by solving a *maximization* problem
- However, (Nash equilibrium) = (Solution to maximization problem) *may not hold in general.* (D, R) does not maximize the payoffs for either player but is still a Nash equilibrium of the game in the example above.
- Goal of this lecture: look at a class of games in which a Nash equilibrium can be found by maximization of a common function related to the payoffs of each player $\rightarrow$ "common function" = potential function
- Games of such class $\rightarrow$ potential games
- Today's topic (and the next two lectures that follow) typically not mentioned in game theory textbooks

II. Exact Potential Games (Monderer and Shapley (1996)[1])

- Potential games are games in strategic form that are in some way equivalent to a game with common payoffs.
- The interpretation of the phrase "in some way" $\rightarrow$ many types of potential games.

[1] Monderer and Shapley (1996) attributes Rosenthal (1973) for the introduction of the concept of potential in strategic form games in the class of what Rosenthal calls *congestion games*. However, the term "potential" was not explicitly used in Rosenthal's paper.

A game G is an **exact potential game** or simply **a potential game** if there exists a function $P : S \to \mathbb{R}$ such that for each $i \in N$ and $s_i, s_i' \in S_i$ and $s_{-i} \in S_{-i}$

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = P(s_i, s_{-i}) - P(s_i', s_{-i})$$

The function P is said to be a **potential function** of the game G.

- A finite potential game always possesses a Nash equilibrium. Also, if each S_i is compact and u_i a continuous function, such an exact potential game also possesses a Nash equilibrium.
- Example – Prisoner's dilemma (reproduced below)

$1 \setminus 2$	C	D
C	$-2, -2$	$-6, 0$
D	$0, -6$	$-5, -5$

- Prisoner's dilemma game is a potential game. The potential function is summarized in the following table.

$1 \setminus 2$	C	D
C	-3	-1
D	-1	0

$$P(C, C) = -3, P(C, D) = P(D, C) = -1, P(D, D) = 0$$

- Note that the potential function is maximized at (D, D), which is a Nash equilibrium in the original game.
- The prisoner's dilemma is a counterexample to the following conjecture: "A strategy combination that maximizes the potential function is optimal for both players."
- If for each $i \in N$, $S_i \subset \mathbb{R}$ is an open set and u_i is a continuously differentiable function on $\prod_{i \in N} S_i \subset \mathbb{R}^n$, then the following is an equivalent condition for a game to be a potential game.
- In the definition, let $s_i' = s_i + h$ for some $h \in \mathbb{R}$ with $h \neq 0$. Then,

$$u_i(s_i + h, s_{-i}) - u_i(s_i, s_{-i}) = P(s_i + h, s_{-i}) - P(s_i, s_{-i})$$

Divide both sides by h and take the limit of $h \to 0$.

Lemma 1. *Let G be a game as described above with for each $S_i \subset \mathbb{R}$ for each $i \in N$ and each u_i is a continuously differentiable function on $\mathbb{R}^n$. Then, the function P is a potential function for the game G if and only if*

$$\frac{\partial u_i}{\partial s_i} = \frac{\partial P}{\partial s_i}$$

Theorem 1. *Suppose in addition that each u_i is twice continuously differentiable. Then, a game G is a potential game if and only if*

$$\frac{\partial^2 u_i}{\partial s_i \partial s_j} = \frac{\partial^2 u_j}{\partial s_i \partial s_j}$$

- Cournot duopoly (with negative prices) – Consider the Cournot duopoly game with prices given simply by the formula, $p(s_1, s_2) = a - (s_1 + s_2)$.
  - The payoff of firm i ($i = 1, 2$) is given by

$$u_i(s_1, s_2) = p(s_1, s_2)s_i - c_i s_i$$

  - The following is a potential function (as shown in Monderer and Shapley (1996))

$$P(s_1, s_2) = a(s_1 + s_2) - (s_1^2 + s_2^2) - s_1 s_2 - (c_1 s_1 + c_2 s_2)$$

III. Exact Potential Games=(Common Payoff Game)+(Dummy Game)

- A game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is a **dummy game** if for each $i \in N$ and $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}) \ \forall s_i, s_i' \in S_i$$

  That is, u_i does not depend on the strategy choice of player i. Therefore, u_i is a function of s_{-i} only.

- Below is a result from Slade (1994), Facchini, van Megen, Borm, and Tijs (1997), and Ui (2000) that gives an alternative characterization of an exact potential game.

Proposition 1. *A game $G = (N, (S_i, u_i)_{i \in N})$ is a potential game $\Leftrightarrow$ there exist functions $P : \prod_{i \in N} S_i \to \mathbb{R}$ and $Q_i : S_{-i} \to \mathbb{R}$ for each $i \in N$ such that for all $s \in \prod_{i \in N} S_i$,*

$$u_i(s) = P(s) + Q_i(s_{-i})$$

- The prisoner's dilemma game can be obtained as the sum of the following two games:
  - First game – common payoff game with the payoffs given the potential function given earlier
  - Second game – dummy game,

$1 \setminus 2$	C	D
C	$-3, -3$	$-1, -1$
D	$-1, -1$	$0, 0$

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$1 \setminus 2$	C	D
C	$1, 1$	$-5, 1$
D	$1, -5$	$-5, -5$

IV. Properties of Potential Games

- Uniqueness up to a constant

Proposition 2. *If P and P' are potential functions corresponding to a potential game G, then for each $s \in \prod_{i \in N} S_i$, $P(s) - P'(s)$ is a constant.*

- For the prisoner's dilemma, the following is also a potential function:

$1 \setminus 2$	C	D
C	0	2
D	2	3

- Improving path
  - A **path** is a (possibly infinite) collection of strategy profiles $(y^0, y^1, \cdots, y^k, \cdots)$ such that for each $k \geq 1$, there exists a unique player $i(k)$ such that $y^k =$

$(x, y^{k-1}_{-i(k)})$ for some $x \in S_{i(k)}$ and $x \neq y^{k-1}_{i(k)}$. $y^k_{-i(k)} := (y^k_j)_{j \neq i(k)}$. That is, each k, there exists a unique player $i(k)$ that changes the strategy being played. All other players choose the same strategy that is chosen in step $k-1$.

– An **improving path** is a path $(y^0, y^1, \cdots, y^k, \cdots)$ such that for each k and the unique player $i(k)$ defined above, $u_{i(k)}(y^{k-1}) < u_{i(k)}(y^k)$.

– A game G has the **finite improvement property (FIP)** if for every improvement path is finite.

Fact 1. *Suppose that a finite game G is a potential game. Then, the following statements hold.*

1. *G satisfies FIP.*
2. *If γ is an improving path (which is finite) and y^0 and y^K are the initial and terminal strategy profiles of γ, then $P(y^0) < P(y^K)$.*

- One characterization using the concept of closed paths.
  – Let γ be a finite path $\gamma = (y^0, y^1, \cdots, y^K)$. Define $I(\gamma)$ to be

$$I(\gamma) = \sum_{k=0}^{K-1} \Big(u_{i(k+1)}(y^{k+1}) - u_{i(k+1)}(y^k) \Big)$$

  – That is, $I(\gamma)$ represents the total change in payoffs from the path γ.

  – A path $\gamma = (y^0, y^1, \cdots, y^K)$ is said to be **closed** if $y^0 = y^K$.

  – A path $\gamma = (y^0, y^1, \cdots, y^K)$ is a **simple closed path** if it is closed and for every $0 \le l \neq k \le K-1$, $y^l \neq y^k$.

  – The **length** of a simple closed path $\gamma = (y^0, y^1, \cdots, y^K)$ is the number of distinct strategy profiles. In this case, the length of γ is K.

Theorem 2. *The following are equivalent.*

1. *Game G is a potential game.*
2. *For every closed path γ, $I(\gamma) = 0$.*

> *3. For every simple closed path γ, $I(\gamma) = 0$.*
>
> *4. For every simple closed path γ of length 4, $I(\gamma) = 0$.*

- Hino (2011) gives a (computationally) simpler equivalent condition of condition 4 in the previous theorem.

V. Weaker Notions of Potential Functions

- Other potential games in Monderer and Shapley (1996):

> A game G is a **weighted potential game** if there exists a function $P : S \to \mathbb{R}$ and a positive weight vector $(w_i)_{i \in N}$ such that for each $i \in N$ and $s_i, s_i' \in S_i$ and $s_{-i} \in S_{-i}$
>
> $$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = w_i(P(s_i, s_{-i}) - P(s_i', s_{-i}))$$

> A game G is an **ordinal potential game** if there exists a function $P : S \to \mathbb{R}$ such that for each $i \in N$ and $s_i, s_i' \in S_i$ and $s_{-i} \in S_{-i}$
>
> $$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) > 0 \ \Leftrightarrow P(s_i, s_{-i}) - P(s_i', s_{-i}) > 0$$

- The next set of potential games are defined using the best response correspondence β_i for each player $i \in N$. The best response potential is defined in Voorneveld (2000), and the pseudo potential is defined in Dubey, Haimanko, and Zapechelnyuk (2006).

> A game G is a **best-response potential game** if there exists a function $P : S \to \mathbb{R}$ such that for each $i \in N$ and $s_{-i} \in S_{-i}$
>
> $$\beta_i(s_{-i}) = \arg\max_{s_i \in S_i} P(s_i, s_{-i})$$

A game G is a **pseudo potential game** if there exists a function $P : S \to \mathbb{R}$ such that for each $i \in N$ and $s_{-i} \in S_{-i}$

$$\beta_i(s_{-i}) \supseteq \arg\max_{s_i \in S_i} P(s_i, s_{-i})$$

where $\arg\max_{s_i \in S_i} P(s_i, s_{-i}) = \{s_i^* \in S_i : P(s_i^*, s_{-i}) \geq P(s_i', s_{-i}) \; \forall s_i' \in S_i\}$

- Some facts regarding these potential games:
  - The relationship among these potential games: Exact potential game $\Rightarrow$ Weighted potential game $\Rightarrow$ Ordinal potential game $\Rightarrow$ Best-reponse potential game $\Rightarrow$ Pseudo potential game
  - Let P be a best-response potential and consider a game $G' = (N, (S_i)_{i \in N}, P)$ where each player's payoff function in G' is the same function P. Then s^* is a Nash equilibrium of G' $\Leftrightarrow$ s^* is a Nash equilibrium of G.
  - Let P be a pseudo potential of a game G. If s^* maximizes P, then s^* is a Nash equilibrium.
- Further potential games: iterated potential (Oyama and Tercieux (2009)), nested potential (Uno (2007)).

VI. Applications

- Techniques used in providing an alternative pricing scheme for economies with externalities $\to$ Sandholm (2002), Sandholm (2005), Sandholm (2007)
- The maximizer of the potential function is a Nash equilibrium that is robust to small changes in information as defined in Kajii and Morris (1997) (Ui (2001))
- Voting (Yamamura and Kawasaki (2013))
- Broadcast games (Kawase and Makino (2013))
- Spatial economics (Oyama (2009), Fujishima (2013))
- Control (Marden, Arslan, and Shamma (2009), Wasa, Hatanaka, and Fujita (2014))

References

Dubey, P., O. Haimanko, and A. Zapechelnyuk (2006). Strategic complements and substitutes, and potential games. *Games and Economic Behavior 54*, 77–94.

Facchini, G., F. van Megen, P. Borm, and S. Tijs (1997). Congestion models and weighted Bayesian potential games. *Theory and Decision 42*, 193–206.

Fujishima, S. (2013). Evolutionary implementation of optimal city size distributions. *Regional Science and Urban Economics 43*, 404–410.

Hino, Y. (2011). An improved algorithm for detecting potential games. *International Journal of Game Theory 40*, 199–205.

Kajii, A. and S. Morris (1997). The robustness of equilibria to incomplete information. *Econometrica 65*, 1283–1310.

Kawase, Y. and K. Makino (2013). Nash equilibria with minimum potential in undirected broadcast games. *Theoretical Computer Science 482*, 33–47.

Marden, J. R., G. Arslan, and J. S. Shamma (2009). Cooperative control and potential games. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics) 39*(6), 1393–1407.

Monderer, D. and L. S. Shapley (1996). Potential games. *Games and Economic Behavior 14*, 124–143.

Oyama, D. (2009). Agglomeration under forward-looking expectations: Potentials and global stability. *Regional Science and Urban Economics 39*, 696–713.

Oyama, D. and O. Tercieux (2009). Iterated potential and robustness of equilibria. *Journal of Economic Theory 144*, 1726–1769.

Rosenthal, R. W. (1973). A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory 2*, 65–67.

Sandholm, W. (2002). Evolutionary implementation and congestion pricing. *Review of Economic Studies 69*, 667–689.

Sandholm, W. (2005). Negative externalities and evolutionary implementation. *Review of Economic Studies 72*, 885–915.

Sandholm, W. (2007). Pigouvian pricing and stochastic evolutionary implementation. *Journal of Economic Theory 132*, 367–382.

Slade, M. (1994). What does an oligopolist maximize? *Journal of Industrial Economics 42*, 45–61.

Ui, T. (2000). A Shapley value representation of potential games. *Games and Economic Behavior 31*, 121–135.

Ui, T. (2001). Robust equilibria of potential games. *Econometrica 69*, 1373–1380.

Uno, H. (2007). Nested potential games. *Economics Bulletin 3*, 1–8.

Voorneveld, M. (2000). Best-response potential games. *Economics Letters 66*, 289–295.

Wasa, Y., T. Hatanaka, and M. Fujita (2014). Application of irrational decisions to simple experimentation to guarantee welfare maximization. *SICE Journal of Control, Measurement, and System Integration 7*(4), 199–204.

Yamamura, H. and R. Kawasaki (2013). Generalized average rules as stable Nash mechanisms to implement generalized median rules. *Social Choice and Welfare 40*, 815–832.