Advanced Noncooperative Game Theory Problem Set 2 (due June 26)

1. For the strategic form game below given in matrix form where the set of players is $N = \{1, 2\}$, find all possible outcomes that can result from the iterated removal of weakly dominated strategies using Version 2W.

$1 \setminus 2$	L	C	R
U	$2, 2$	$1, 3$	$-1, 3$
M	$1, -1$	$1, 3$	$0, 3$
D	$0, 4$	$0, 1$	$0, 0$

2. In Lecture 2, it was argued that two versions of iterated removal of strictly dominated strategies (Version1 and Version 2) yield the same result as long as S_i is finite. Now, consider the following game with $N = \{1, 2\}$, $S_i = \{0, 1, 2, \cdots\}$. Let the payoff functions u_i be defined as follows. For $s = (s_1, s_2)$ with $s_1 \neq 0$ and $s_2 \neq 0$, $u_i(s_i, s_j) = s_i$ for $i = 1, 2$ and $j \neq i$. If $s_1 = 0$, then define $u_1(s_1, s_2) = u_2(s_1, s_2) = s_2$, and if $s_2 = 0$, let $u_1(s_1, s_2) = u_2(s_1, s_2) = s_1$.

   (a) Show that strategy 0 is the only strategy that is not strictly dominated.

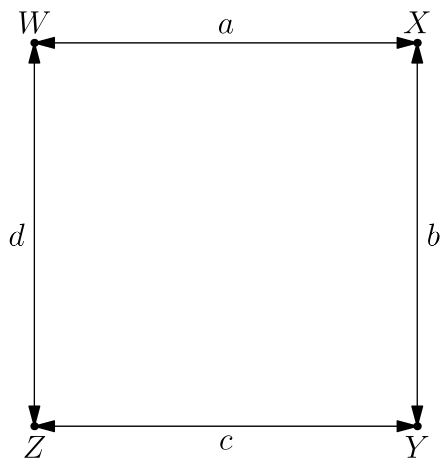
   (b) Show that Version 1 and Version 2 of the iterated removal of strictly dominated strategies do not yield the same result.

   (c) Consider another version in which "any number of strategies that are strictly dominated can be eliminated." Show that this version yields a different result from Version 1 and Version 2.

   (d) Determine whether $(0, 0)$ is a Nash equilibrium or not.

3. Let $N = \{1, 2\}$ be the set of players, and consider the following "congestion" game. Player 1 who is at point W needs to get to point Y, while player 2 needs to get from point X to Z (see figure below). Relabel the roads in the following way:

   - Label (unordered) edge WX as road a.
   - Label (unordered) edge XY as road b.
   - Label (unordered) edge YZ as road c.
   - Label (unordered) edge WZ as road d.



Let $v_j(k)$ denote the payoff from k players using road j. Assign the following values:

$$v_a(1) = 12, v_b(1) = 8, v_c(1) = 6, v_d(1) = 9.$$

$$v_a(2) = 6, v_b(2) = 5, v_c(2) = 4, v_d(2) = 3.$$

The payoff from selecting a particular route is defined by the sum of the payoffs derived from each road. To simply the problem, suppose that player 1's strategies are to choose route (a, b) or route (d, c). Similarly player 2's strategies are to choose route (b, c) or route (a, d).

(a) Formulate the game in strategic form by using a matrix form.

(b) Find all pure strategy Nash equilibria.

4. Use the result in Question 1 and Question 2 of Problem Set 1 to show the following statements.

   (a) Let $\sigma_i \in \Delta(S_i)$ be a best response to $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$. Then, every $s_i \in S_i$ such that $\sigma_i(s_i) > 0$ is a best response to σ_{-i}.

   (b) Let $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$, and consider a mixed strategy $\sigma_i \in \Delta(S_i)$ where $\sigma_i(s_i) > 0$ for all $s_i \in S_i$ such that s_i is a best response to σ_{-i}. Show that σ_i is also a best response to σ_{-i}.

   (c) Using the two previous parts, prove the following statement.

The mixed strategy profile $\sigma^* \in \prod_{i \in N} \Delta(S_i)$ is a Nash equilibrium if and only if for every $i \in N$ and every $s_i, s_i' \in S_i$ with $\sigma_i^*(s_i) > 0$, $\sigma_i^*(s_i') > 0$ and any $\bar{s}_i \in S_i$ with $\sigma_i^*(\bar{s}_i) = 0$,

$$\pi_i(s_i, \sigma_{-i}^*) = \pi_i(s_i', \sigma_{-i}^*)$$

$$\pi_i(s_i, \sigma_{-i}^*) \geq \pi_i(\bar{s}_i, \sigma_{-i}^*)$$

5. Consider a situation with two players deciding how to split a payoff of 100. Let $N = \{1, 2\}$ be the set of players and $S_1 = S_2 = [0, 100]$ be the set of strategies denoting the amount of payoff each player demands. For $s_1, s_2 \in [0, 100]$, if $s_1 + s_2 \leq 100$, then player 1 receives s_1, while player 2 receives s_2. If $s_1 + s_2 > 100$, then both player receive a payoff of 0.

   (a) Give an explicit formulation of each player i's payoff function u_i, and show that it is not continuous.

   (b) Find all Nash equilibria in pure strategies of this game.