Advanced Noncooperative Game Theory Problem Set 1 (due June 19)

1. Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game in strategic form where S_i is finite for all $i \in N$ and consider its mixed extension, $(N, (\Delta(S_i))_{i\in N}, (\pi_i)_{i\in N})$. Denote by $\pi_i(s_i, \sigma_{-i})$ the expected payoff when player i chooses pure strategy $s_i \in S_i$ and the other players $j \neq i$ choose mixed strategies $\sigma_j \in \Delta(S_j)$. Let $\sigma_i \in \Delta(S_i)$ be any mixed strategy. Using the definition of π_i in the lecture notes, show that the following hold:

   (a)
$$\pi_i(s_i, \sigma_{-i}) = \sum_{s_{-i}\in S_{-i}} \left(\prod_{j\neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i})$$

   (b)
$$\pi_i(\sigma_i, \sigma_{-i}) = \sum_{s_i\in S_i} \sigma_i(s_i)\pi_i(s_i, \sigma_{-i}).$$

2. Consider the same setup as Question 1. Let $\sigma_{-i} \in \prod_{j\neq i} \Delta(S_j)$ be a profile of mixed strategies for players $j \neq i$, and for a mixed strategy σ_i of player i, let $C(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}$ be the set of pure strategies that σ_i places a positive probability on. Then, using Question 1 part (b), show that the following inequality holds.
$$\pi(\sigma_i, \sigma_{-i}) \leq \max_{s_i\in C(\sigma_i)} \pi(s_i, \sigma_{-i})$$

3. Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game in strategic form. Suppose that $s_i \in S_i$ is strictly dominated by $s_i' \in S_i$, and s_i' is strictly dominated by $s_i'' \in S_i$.

   (a) Show that s_i is strictly dominated by s_i''.

   (b) Is the statement in part (a) true if "strictly dominated" is replaced by "weakly dominated?" If the statement is true, give a proof of the modified statement; if it is false, then provide a counterexample.

4. (Bertrand Duopoly) Suppose there are two firms producing the same good and each firm has the same technology in that it costs $c > 0$ for each unit produced. Each firm chooses (independently and simultaneously) a price out of the set of all positive real numbers. The firm with the lower price can then produce an amount that is decided by the demand function D, so that if a firm that chose p would have to produce $D(p)$ units. The firm with the higher price does not produce any units and is left out of the market. If both firms choose the same price, a coin is flipped that

gives each firm a probability of $1/2$ of winning, and the winner supplies the market, while the loser does not. Suppose that $D(p) = \max\{a - p, 0\}$ represents the market demand as a function of p where $a > c$.

(a) Formulate this situation as a game in strategic form.

(b) Does each firm have a strictly dominated strategy? If so, find all such strategies.

(c) Does each firm have a weakly dominated strategy? If so, find all such strategies.